(T1) **True or False**
Determine if each of the following statements is True or False. In the Summary Answersheet, tick the correct answer (TRUE / FALSE) for each statement. No justifications are necessary for this question.

(T1.1) In a photograph of the clear sky on a Full Moon night with a sufficiently long exposure, the colour of the sky would appear blue as in daytime. **2**

Solution:
T
The colour of the clear sky during night is the same as during daytime, since the spectrum of sunlight reflected by the Moon is almost the same as the spectrum of sunlight. Only the intensity is lower.
2.0

(T1.2) An astronomer at Bhubaneswar marks the position of the Sun on the sky at 05:00 UT every day of the year. If the Earth's axis were perpendicular to its orbital plane, these positions would trace an arc of a great circle. **2**

Solution:
T
If the Earth's axis were perpendicular to its orbital plane, the celestial equator will coincide with ecliptic and the Sun will remain along the celestial equator every day. However, as the Earth's orbit is elliptical, the true sun would still lead or lag mean sun by a few minutes on different days of year.
2.0

(T1.3) If the orbital period of a certain minor body around the Sun in the ecliptic plane is less than the orbital period of Uranus, then its orbit must necessarily be fully inside the orbit of Uranus. **2**

Solution:
F
The semi-major axis of the orbit of the body will be less than that of Uranus. However the minor body's orbit may have a high eccentricity, in which case it may go outside that of Uranus.
2.0

(T1.4) The centre of mass of the solar system is inside the Sun at all times. **2**

Solution:
F
The centre of mass of Sun-Jupiter pair is just outside the Sun. Thus, if all gas giants are on same side of the Sun, the centre of mass of Solar system is definitely outside the Sun.
2.0

(T1.5) A photon is moving in free space. As the Universe expands, its momentum decreases. **2**

Solution:
T
For photons the wavelength increases when the Universe expands.
2.0

(T2) **Gases on Titan** **10**

Gas particles in a planetary atmosphere have a wide distribution of speeds. If the r.m.s. (root mean square) thermal speed of particles of a particular gas exceeds 1/6 of the escape speed, then most of that gas will escape from the planet. What is the minimum atomic weight (relative atomic mass), $A_{\rm min}$, of an ideal monatomic gas so that it remains in the atmosphere of Titan?

Given, mass of Titan $M_{\rm T} = 1.23 \times 10^{23}$ kg, radius of Titan $R_{\rm T} = 2575$ km, surface temperature of Titan $T_{\rm T} = 93.7$ K.

Solution:

As the gas is monatomic,

$$\frac{3}{\not 2} k_{\rm B} T_{\rm T} \approx \frac{1}{\not 2} m_{\rm g} v_{\rm rms}^2$$
$$3 k_{\rm B} T_{\rm T} \approx \frac{M_{\rm g}}{N_{\rm A}} v_{\rm rms}^2$$
$$\therefore v_{\rm rms} \approx \sqrt{\frac{3 k_{\rm B} N_{\rm A} T_{\rm T}}{M_{\rm g}}} \qquad \textbf{4.0}$$

50% **deduction if** 3/2 **pre-factor is not used and** 1/2 **or** 1 **are used instead.**
Full credit if students writes the relation for $v_{\rm rms}$ directly.

To remain in atmosphere,

$$v_{\rm rms} < \frac{v_{\rm esc}}{6} = \frac{1}{6}\sqrt{\frac{2GM_{\rm T}}{R_{\rm T}}}$$
$$\sqrt{\frac{3 k_{\rm B} N_{\rm A} T_{\rm T}}{M_{\rm g}}} < \sqrt{\frac{GM_{\rm T}}{18 R_{\rm T}}}$$
$$\therefore M_{\rm g} > \frac{54 k_{\rm B} N_{\rm A} T_{\rm T} R_{\rm T}}{GM_{\rm T}} \qquad \textbf{4.0}$$
$$> \frac{54 \times 1.381 \times 10^{-23} \times 6.022 \times 10^{23} \times 93.7 \times 2.575 \times 10^6}{6.6741 \times 10^{-11} \times 1.23 \times 10^{23}}\ \text{g}$$
$$> 13.2\ \text{g} \qquad \textbf{1.5}$$

Thus, all gases with atomic weight more than $\boxed{A_{\rm min} = 13.2}$ will be retained in the atmosphere of Titan. **0.5**

Half mark for understanding that atomic mass has no units.

<u>Alternative solution</u>

$$\frac{3}{\not 2} k_{\rm B} T_{\rm T} \approx \frac{1}{\not 2} m_{\rm g} v_{\rm rms}^2$$
$$\therefore v_{\rm rms} \approx \sqrt{\frac{3 k_{\rm B} T_{\rm T}}{m_{\rm g}}} \qquad \textbf{4.0}$$
$$\therefore m_{\rm g} > \frac{54 k_{\rm B} T_{\rm T} R_{\rm T}}{GM_{\rm T}} \qquad \textbf{4.0}$$
$$m_{\rm g} > 2.19 \times 10^{-26}\ \text{kg} \qquad \textbf{1.0}$$
$$\therefore A_{\rm min} = \frac{m_{\rm g}}{\text{atomic mass unit}} = \frac{2.19 \times 10^{-26}\ \text{kg}}{1.66 \times 10^{-27}\ \text{kg}}$$

$\boxed{A_{\min} = 13.2}$ **1.0**

Answers between 13.0 and 13.4 are acceptable with full credit.

(T3) **Early Universe**

Cosmological models indicate that radiation energy density, ρ_r, in the Universe is proportional to $(1+z)^4$, and the matter energy density, ρ_m, is proportional to $(1+z)^3$, where z is the redshift. The dimensionless density parameter, Ω, is given as $\Omega = \rho/\rho_c$, where ρ_c is the critical energy density of the Universe. In the present Universe, the density parameters corresponding to radiation and matter, are $\Omega_{r_0} = 10^{-4}$ and $\Omega_{m_0} = 0.3$, respectively.

(T3.1) Calculate the redshift, z_e, at which radiation and matter energy densities were equal. **[3]**

Solution:

$$\frac{\rho_{m_0}/\rho_c}{\rho_{r_0}/\rho_c} = \frac{\Omega_{m_0}}{\Omega_{r_0}} = \frac{0.3}{10^{-4}} = 3000$$

At z_e, both matter density and radiation density were equal.

$$\rho_r = \rho_m$$

$$\therefore \rho_{r_0}(1+z_e)^4 = \rho_{m_0}(1+z_e)^3$$

$$1 + z_e = \frac{\rho_{m_0}}{\rho_{r_0}} = 3000 \qquad \textbf{2.0}$$

$$\therefore z_e \simeq \boxed{3000} \qquad \textbf{1.0}$$

Only $z_e = 2999$ and $z_e = 3000$ are acceptable answers.

(T3.2) Assuming that the radiation from the early Universe has a blackbody spectrum with a temperature of 2.732 K, estimate the temperature, T_e, of the radiation at redshift z_e. **[4]**

Solution:

As the Universe behaves like an ideal black body, the radiation density will be proportional to the fourth power of the temperature (Stefan's law).

$$\left(\frac{T_e}{T_0}\right)^4 = \frac{\rho_{r_e}}{\rho_{r_0}}$$

$$= \frac{\rho_{r_0}(1+z_e)^4}{\rho_{r_0}} \qquad \textbf{2.0}$$

$$\left(\frac{T_e}{2.732}\right)^4 = (1+z_e)^4 \qquad \textbf{1.0}$$

$$\frac{T_e}{2.732} = 1 + z_e = 3000$$

$$T_e = 3000 \times 2.732$$

$$T_e = \boxed{8200\,\mathrm{K}} \qquad \textbf{1.0}$$

$8100 \leq T_e \leq 8200$ **gives 1.0;** $8200 < T_e \leq 9000$ **gives 0.5; else 0.**

(T3.3) Estimate the typical photon energy, E_ν (in eV), of the radiation as emitted at redshift z_e. **[3]**

Solution:
Wien's law:

$$\lambda_{\max} = \frac{0.002898\,\mathrm{m\,K}}{T_e}$$
$$= \frac{0.002898}{8200}\,\mathrm{m} = 354\,\mathrm{nm} \qquad \mathbf{1.0}$$
$$E_\nu = \frac{hc}{\lambda_{\max}}$$
$$= \frac{6.62\times10^{-34}\times3\times10^{8}}{354\times10^{-9}}\,\mathrm{J} = \frac{5.62\times10^{-19}}{1.602\times10^{-19}}\,\mathrm{eV} \qquad \mathbf{1.0}$$
$$E_\nu = \boxed{3.5\,\mathrm{eV}} \qquad \mathbf{1.0}$$

Alternative solution:

$$E_\nu = k_\mathrm{B}T_\mathrm{e}$$
$$= 1.38\times10^{-23}\times8200\,\mathrm{J} = \frac{1.13\times10^{-19}}{1.602\times10^{-19}}\,\mathrm{eV} \qquad \mathbf{2.0}$$
$$E_\nu = \boxed{0.71\,\mathrm{eV}} \qquad \mathbf{1.0}$$

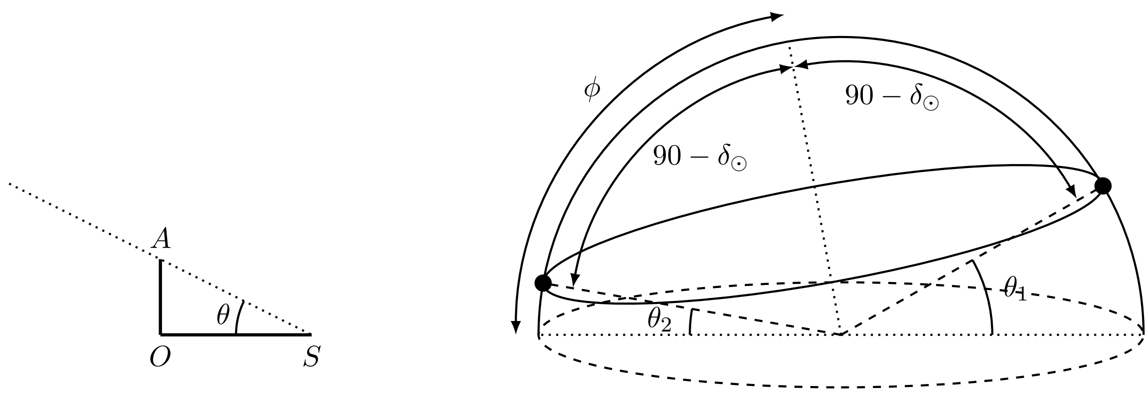
Use of either Wien's law or $E = k_\mathbf{B}T$ gets full credit. $E_\nu = 3k_\mathbf{B}T/2$ or similar gets no credit. Answers with $E_\nu = 3k_\mathbf{B}T$ or $E_\nu = 2.7k_\mathbf{B}T$ also get full credit.

(T4) **Shadows** 10

An observer in the northern hemisphere noticed that the length of the shortest shadow of a 1.000 m vertical stick on a day was 1.732 m. On the same day, the length of the longest shadow of the same vertical stick was measured to be 5.671 m.

Find the latitude, ϕ, of the observer and declination of the Sun, $\delta_\odot$, on that day. Assume the Sun to be a point source and ignore atmospheric refraction.

Solution:
As the longest shadow of the Sun on the given day is of finite length, the Sun is circumpolar for this observer on this day. **2.0**



In the figure above, the left panel shows the shadow OS formed by stick OA (of length 1.000 m), and the right panel shows the Sun's location in two cases.

For an altitude θ of the Sun,

$$\tan\theta = \frac{OA}{OS} = \frac{1.000\,\text{m}}{OS}$$

$$\therefore \cot\theta = OS \text{ (in metres)}$$ **1.0**

Let θ_1 and θ_2 be altitude in two extreme cases.

$$\theta_1 = 180^\circ - \phi - (90^\circ - \delta_\odot) = 90^\circ - \phi + \delta_\odot$$ **1.0**

$$\cot(90^\circ - \phi + \delta_\odot) = 1.732$$

$$\therefore \tan(\phi - \delta_\odot) = 1.732$$

$$\phi - \delta_\odot = \tan^{-1}(1.732) = 60^\circ = 1.047\,\text{rad}$$ **1.5**

$$\theta_2 = \phi - (90^\circ - \delta_\odot) = \phi - 90^\circ + \delta_\odot$$ **1.0**

$$\cot(\phi - 90^\circ + \delta_\odot) = 5.671$$

$$\therefore \tan(\phi - 90^\circ + \delta_\odot) = \frac{1}{5.671}$$

$$\phi + \delta_\odot = \tan^{-1}\left(\frac{1}{5.671}\right) + 90^\circ = 100^\circ = 1.745\,\text{rad}$$ **1.5**

Solving,

$$\boxed{\phi = 80^\circ} = 1.396\,\text{rad}$$ **1.0**

$$\boxed{\delta_\odot = 20^\circ} = 0.349\,\text{rad}$$ **1.0**

Given high accuracy of shadow length, only $\pm 0.5^\circ$ is allowed.

One can also solve the question by manipulating $\tan(\phi - \delta_\odot)$ and $\tan(\phi + \delta_\odot)$, to get $\tan(\phi)$ and $\tan(\delta_\odot)$.

(T5) **GMRT beam transit** **[10]**

Giant Metrewave Radio Telescope (GMRT), one of the world's largest radio telescopes at metre wavelengths, is located in western India (latitude: $19^\circ 6'$ N, longitude: $74^\circ 3'$ E). GMRT consists of 30 dish antennas, each with a diameter of 45.0 m. A single dish of GMRT was held fixed with its axis pointing at a zenith angle of $39^\circ 42'$ along the northern meridian such that a radio point source would pass along a diameter of the beam, when it is transiting the meridian.

What is the duration T_{transit} for which this source would be within the FWHM (full width at half maximum) of the beam of a single GMRT dish observing at 200 MHz?

Hint: The FWHM size of the beam of a radio dish operating at a given frequency corresponds to the angular resolution of the dish. Assume uniform illumination.

Solution:

As the dish is pointed towards northern meridian at zenith angle of 39.7°, altitude of the centre of the beam is

$$a = 90.00^\circ - z = 90.00^\circ - 39.70^\circ = 50.30^\circ$$ **1.0**

Thus, declination of the source should be,

$$\delta = 90.00 - a + \phi = 90.00^\circ - 50.30^\circ + 19.10^\circ = 58.80^\circ$$ **2.0**

Declination = ZA + Latitude also gets full credit.

FWHM beam size (for uniform illumination) will be given by

$$\theta = \frac{1.22\lambda}{D}$$ **1.5**

$$= \frac{1.22c}{D\nu} = \frac{1.22 \times 2.998 \times 10^8}{45.0 \times 2 \times 10^8}$$

$$= 0.0406\,\text{rad}$$

$$\theta = 2.33^\circ$$ **1.5**

$$T_{\text{transit}} = \frac{\theta \times 3.99\,\text{min}}{\cos\delta}$$ **3.0**

$$= \frac{2.33 \times 3.99\,\text{min}}{\cos 58.8}$$

$$\boxed{T_{\text{transit}} = 17.9\,\text{min}}$$ **1.0**

- **Use of 4 min per degree is also acceptable.**
- **Missing $\cos\delta$ gets a penalty of 2.0.**

(T6) **Cepheid Pulsation**

The star β-Doradus is a Cepheid variable star with a pulsation period of 9.84 days. We make a simplifying assumption that the star is brightest when it is most contracted (radius being R_1) and it is faintest when it is most expanded (radius being R_2). For simplicity, assume that the star maintains its spherical shape and behaves as a perfect black body at every instant during the entire cycle. The bolometric magnitude of the star varies from 3.46 to 4.08. From Doppler measurements, we know that during pulsation the stellar surface expands or contracts at an average radial speed of $12.8\,\text{km}\,\text{s}^{-1}$. Over the period of pulsation, the peak of thermal radiation (intrinsic) of the star varies from 531.0 nm to 649.1 nm.

(T6.1) Find the ratio of radii of the star in its most contracted and most expanded states (R_1/R_2). **[7]**

Solution:

We first find flux ratio and then use Stefan's law to compare the fluxes.

$$m_1 - m_2 = -2.5 \log\left(\frac{F_1}{F_2}\right)$$ **1.0**

$$\therefore \frac{F_1}{F_2} = 10^{-0.4(m_1 - m_2)} = 10^{-0.4(3.46-4.08)}$$

$$= 1.77$$ **1.0**

$$L_i = 4\pi R_i^2 \sigma T_i^4$$ **1.0**

$$\therefore F_1 = \frac{4\pi R_1^2 \sigma T_1^4}{4\pi D^2} \qquad F_2 = \frac{4\pi R_2^2 \sigma T_2^4}{4\pi D^2}$$ **1.0**

$$\frac{F_1}{F_2} = \frac{R_1^2}{R_2^2} \times \frac{T_1^4}{T_2^4}$$
$$\frac{R_1}{R_2} = \sqrt{\frac{F_1}{F_2}} \times \left(\frac{T_2}{T_1}\right)^2$$

From Wien's displacement law, $\frac{T_2}{T_1} = \frac{\lambda_1}{\lambda_2}$. **1.0**

$$\therefore \frac{R_1}{R_2} = \sqrt{\frac{F_1}{F_2}} \times \left(\frac{\lambda_1}{\lambda_2}\right)^2$$ **1.0**

$$= \sqrt{1.77} \times \left(\frac{531.0}{649.1}\right)^2$$

$$\boxed{\frac{R_1}{R_2} = 0.890}$$ **1.0**

Acceptable range: ± 0.010.

(T6.2) Find the radii of the star (in metres) in its most contracted and most expanded states (R_1 and R_2). **[3]**

Solution:

$$R_2 - R_1 = v \times P/2$$ **2.0**

$$R_2 - R_1 = 12.8 \times 10^3 \times 86\,400 \times \frac{9.84}{2}\,\mathrm{m}$$

$$(1 - 0.890)R_2 = 5.441 \times 10^9\,\mathrm{m}$$

$$\therefore \boxed{R_2 = 4.95 \times 10^{10}\,\mathrm{m}}$$ **0.5**

$$\boxed{R_1 = 4.41 \times 10^{10}\,\mathrm{m}}$$ **0.5**

Acceptable range: $\pm 0.02 \times 10^{10}\,\mathrm{m}$ **for both.**

(T6.3) Calculate the flux of the star, F_2, when it is in its most expanded state. **[5]**

Solution:
To get the absolute value of flux (F_2) we must compare it with observed flux of the Sun.

$$m_2 - m_\odot = -2.5 \log\left(\frac{F_2}{F_\odot}\right)$$

$$\therefore F_2 = F_\odot 10^{-0.4(m_2 - m_\odot)}$$ **3.0**

$$= \frac{L_\odot}{4\pi a_\oplus^2} \times 10^{-0.4(4.08+26.72)}$$

$$= \frac{3.826 \times 10^{26} \times 4.7863 \times 10^{-13}}{4\pi(1.496 \times 10^{11})^2}\,\mathrm{W\,m^{-2}}$$

$$\boxed{F_2 = 6.51 \times 10^{-10}\,\mathrm{W\,m^{-2}}}$$ **2.0**

Acceptable range: $\pm 0.04 \times 10^{-10}\,\mathrm{W\,m^{-2}}$.

(T6.4) Find the distance to the star, D_{star}, in parsecs. **5**

Solution:

$$D_{\text{star}} = \sqrt{\frac{L_2}{4\pi F_2}} = \sqrt{\frac{R_2^2 \sigma T_2^4}{F_2}} = R_2 T_2^2 \sqrt{\frac{\sigma}{F_2}}$$ **2.0**

Wien's law: $T_2 = \dfrac{2.898 \times 10^{-3}\,\text{m K}}{\lambda_2}$ **1.0**

$$D_{\text{star}} = 4.95 \times 10^{10} \times \left(\frac{2.898 \times 10^{-3}}{649.1 \times 10^{-9}}\right)^2 \sqrt{\frac{5.670 \times 10^{-8}}{6.51 \times 10^{-10}}}$$

$$\therefore \boxed{D_{\text{star}} = 9.208 \times 10^{18}\,\text{m} = 298\,\text{pc}}$$ **2.0**

Acceptable range: 298 ± 2pc **(depends on truncation).**

(T7) **Telescope optics**

In a particular ideal refracting telescope of focal ratio $f/5$, the focal length of the objective lens is 100 cm and that of the eyepiece is 1 cm.

(T7.1) What is the angular magnification, m_0, of the telescope? What is the length of the telescope, L_0, i.e. the distance between its objective and eyepiece? **4**

Solution:

The magnification will be given by,

$$m_0 = \frac{f_o}{f_e}$$ **1.0**

$$= \frac{100}{1} = 100$$ **1.0**

The magnification is $\boxed{m_0 = 100}$

Length of the telescope will be

$$L_0 = f_o + f_e$$ **1.0**

$$= 100 + 1 = 101\,\text{cm}$$ **1.0**

The telescope length will be $\boxed{L_0 = 101\,\text{cm}}$

Exact answer required for credit.

An introduction of a concave lens (Barlow lens) between the objective lens and the prime focus is a common way to increase the magnification without a large increase in the length of the telescope. A Barlow lens of focal length 1 cm is now introduced between the objective and the eyepiece to double the magnification.

(T7.2) At what distance, d_B, from the prime focus must the Barlow lens be kept in order to obtain this desired double magnification? **6**

Solution:

We use the following sign convention. Lens is the origin. Direction along the direction

of light is taken as positive. The lens formula is $\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$ (f is positive for convex lens and negative for concave lens). Magnification is $m = \frac{v}{u}$. Solutions using other sign conventions are acceptable.
Let v be image distance from the Barlow lens.

$$\frac{1}{f_\text{B}} = \frac{1}{v} - \frac{1}{u}$$ **1.0**

Distance of Barlow lens, d_B, before the prime focus is same as the object distance, u, in this case. **1.0**

$$\frac{1}{f_\text{B}} = \frac{1}{v} - \frac{1}{d_B}$$

Also, $m_\text{B} = 2 = \frac{v}{u} = \frac{v}{d_\text{B}}$ **0.5**

$$\therefore \frac{1}{d_\text{B}} = \frac{2}{v}$$ **1.0**

$$\therefore \frac{1}{-1} = \frac{1}{v} - \frac{2}{v}$$

$$-1 = \frac{-1}{v}$$

$$v = 1\,\text{cm}$$

$$d_\text{B} = \frac{v}{2} = \frac{1\,\text{cm}}{2} = 0.5\,\text{cm}$$ **2.5**

The positive sign for d_B indicates that the Barlow lens was introduced $\boxed{0.5\,\text{cm}}$ before the prime focus.

(T7.3) What is the increase, ΔL, in the length of the telescope? **[4]**

Solution:
The increase in the length will be,

$$\Delta L = v - d_\text{B}$$ **2.0**

$$= 1.0 - 0.5 = 0.5\,\text{cm}$$ **2.0**

Thus, the length will be increased by $\boxed{\Delta L = 0.5\,\text{cm}}$

A telescope is now constructed with the same objective lens and a CCD detector placed at the prime focus (without any Barlow lens or eyepiece). The size of each pixel of the CCD detector is 10 µm.

(T7.4) What will be the distance in pixels between the centroids of the images of the two stars, n_p, on the CCD, if they are 20″ apart on the sky? **[6]**

Solution:
Plate scale at prime focus is given by,

$$s = \frac{1}{f_o} = \frac{1\,\text{rad}}{1\,\text{m}} = 0.206\,265\,\text{arcsec}/\mu\text{m}$$ **2.0**

Since each pixel is 10 µm in size,

$$s_\mathrm{p} = 10 \times 0.206\,\mathrm{arcsec/\mu m} = 2.06\,\mathrm{arcsec/pixel}$$ **2.0**

Two stars will be separated by,

$$n_\mathrm{p} = \frac{20''}{2.06''}\ \text{pixels} \simeq \boxed{10\ \text{pixels}}$$ **2.0**

Acceptable range: 9.5 to 10.5 pixels.

(T8) **U-band photometry**

A star has an apparent magnitude $m_U = 15.0$ in the U-band. The U-band filter is ideal, i.e., it has perfect (100%) transmission within the band and is completely opaque (0% transmission) outside the band. The filter is centered at 360 nm, and has a width of 80 nm. It is assumed that the star also has a flat energy spectrum with respect to frequency. The conversion between magnitude, m, in any band and flux density, f, of a star in Jansky ($1\,\mathrm{Jy} = 1 \times 10^{-26}\,\mathrm{W\,Hz^{-1}\,m^{-2}}$) is given by

$$f = 3631 \times 10^{-0.4m}\,\mathrm{Jy}$$

(T8.1) Approximately how many U-band photons, N_0, from this star will be incident normally on a $1\,\mathrm{m}^2$ area at the top of the Earth's atmosphere every second? **8**

Solution:

The U-band is defined as (360 ± 40) nm. Thus, the maximum, minimum and average frequencies of the band will be,

$$\nu_\mathrm{max} = \frac{c}{\lambda_\mathrm{max}} = 9.369 \times 10^{14}\,\mathrm{Hz}$$

$$\nu_\mathrm{min} = 7.495 \times 10^{14}\,\mathrm{Hz}$$

$$\nu_\mathrm{avg} = 8.432 \times 10^{14}\,\mathrm{Hz}$$

$$\Delta\nu = \nu_\mathrm{max} - \nu_\mathrm{min} = 1.874 \times 10^{14}\,\mathrm{Hz}$$ **2.0**

$$f_\mathrm{st1} = 3631 \times 10^{-0.4\times15} = 3.631\,\mathrm{mJy} = 3.631 \times 10^{-29}\,\mathrm{W\,Hz^{-1}\,m^{-2}}$$ **2.0**

$$\text{Now, } N_0 \times h\nu_\mathrm{avg} = \Delta\nu \times f_\mathrm{st1} \times A \times \Delta t$$ **2.0**

where, $A = 1\,\mathrm{m}^2$ & $\Delta t = 1\,\mathrm{s}$

$$\therefore N_0 = \frac{1.874 \times 10^{14} \times 3.631 \times 10^{-29}}{6.626 \times 10^{-34} \times 8.432 \times 10^{14}} \simeq \boxed{12180}$$ **2.0**

Exact calculation including integration is accepted with full credit (exact answer: 12190).

Accepted range: 12180 ± 200.

Using flat spectrum for $\Delta\lambda$ instead of $\Delta\nu$ is considered a major conceptual error, and will incur penalty of 2.0 marks.

This star is being observed in the U-band using a ground based telescope, whose primary mirror has a diameter of 2.0 m. Atmospheric extinction in U-band during the observation is 50%. You may assume that the seeing is diffraction limited. Average surface brightness of night sky in U-band was measured to be 22.0 mag/arcsec^2.

(T8.2) What is the ratio, R, of number of photons received per second from the star to that received from the sky, when measured over a circular aperture of diameter $2''$? **8**

Solution:
Let us call sky flux per square arcsec as Φ and total sky flux for the given aperture as ϕ_{sky}. Let total star flux be ϕ_{st}.

$$\phi_{\text{sky}} = A\Phi = \pi \times (1\,\text{arcsec})^2 \times \Phi = \pi\Phi \qquad \mathbf{3.0}$$

$$\therefore m_{\text{sky}} = 22.0 + 2.5\log_{10}\left(\frac{\Phi}{\phi_{\text{sky}}}\right) \qquad \mathbf{1.0}$$

$$= 22.0 + 2.5\log_{10}\left(\frac{\not{\Phi}}{\pi\not{\Phi}}\right)$$

$$= 22.0 - 2.5\log_{10}(\pi)$$

$$m_{\text{sky}} = 20.76 \text{ mag} \qquad \mathbf{1.0}$$

As extinction is 50% **1.0**

$$R = \frac{\phi_{\text{st2}}}{\phi_{\text{sky}}} = \frac{0.5\phi_{\text{st1}}}{\phi_{\text{sky}}} = 0.5 \times 10^{(20.76-15)/2.5}$$

$$\simeq \boxed{100} \qquad \mathbf{2.0}$$

Accepted range: ±5.
A student may also calculate number of photons incident per second per metre square in this case and then compare it with the answer in the first case to get the correct ratio.

(T8.3) In practice, only 20% of U-band photons falling on the primary mirror are detected. How many photons, N_{t}, from the star are detected per second? **4**

Solution:

$$N_{\text{t}} \times 1\,\text{m}^2 = N_0 \times 0.5 \times 0.2 \times A_{\text{t}} \qquad \mathbf{2.0}$$

$$N_{\text{t}} = 12180 \times 0.5 \times 0.2 \times \pi\left(\frac{2.0}{2}\right)^2 = 1233\pi \qquad \mathbf{1.0}$$

$$N_{\text{t}} \simeq \boxed{3813} \qquad \mathbf{1.0}$$

Accepted range (3813 ± 50)**.**

(T9) **Mars Orbiter Mission**

India's Mars Orbiter Mission (MOM) was launched using the Polar Satellite Launch Vehicle (PSLV) on 5 November 2013. The dry mass of MOM (body + instruments) was 500 kg and it carried fuel of mass 852 kg. It was initially placed in an elliptical orbit around the Earth with perigee at a height of 264 km and apogee at a height of 23 904 km, above the surface of the Earth. After raising the orbit six times, MOM was transferred to a trans-Mars injection orbit (Hohmann orbit).

The first such orbit-raising was performed by firing the engines for a very short time near the perigee. The engines were fired to change the orbit without changing the plane of the orbit and without changing its perigee. This gave a net impulse of $1.73 \times 10^5\,\text{kg}\,\text{m}\,\text{s}^{-1}$ to the satellite. Ignore the change in mass due to burning of fuel.

(T9.1) What is the height of the new apogee, h_{a}, above the surface of the Earth, after this **14** engine burn?

Solution:
Let apogee and perigee distances be r_a and r_p respectively.

$$r_p = R_\oplus + h_p^i = (6371 + 264)\,\text{km} = 6635\,\text{km}$$ **1.0**

$$r_a = R_\oplus + h_a^i = (6371 + 23904)\,\text{km} = 30\,275\,\text{km}$$ **1.0**

Conservation of energy and angular momentum gives

$$E = -\frac{GmM_\oplus}{r_p + r_a}$$

Total energy at perigee

$$\frac{1}{2}m{v_p}^2 - \frac{GmM_\oplus}{r_p} = E = -\frac{GmM_\oplus}{r_p + r_a}$$

$$\frac{1}{2}{v_p}^2 = \frac{GM_\oplus}{r_p}\left(1 - \frac{r_p}{r_p + r_a}\right)$$

$$\therefore v_p = \sqrt{2\frac{GM_\oplus}{r_a + r_p}\frac{r_a}{r_p}}$$ **2.0**

$$= \sqrt{\frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24} \times 3.0275 \times 10^{7}}{6.635 \times 10^{6} \times (3.0275 + 6.635 \times 10^{6})}}$$

$$= 9.929\,\text{km}\,\text{s}^{-1}$$ **1.0**

As the engine burn is just 41.6 s, we assume that the entire impulse is applied instantaneously at perigee. The impulse is $J = 1.73 \times 10^5\,\text{kg}\,\text{m}\,\text{s}^{-1}$. Note that the total mass of MOM must include the fuel, so we have to use
$m = 500 + 852 = 1352\,\text{kg}$. **1.0**
Change in velocity due to impulse at perigee is

$$\Delta v = \frac{J}{m} = \frac{1.73 \times 10^5}{1352} = 128.0\,\text{m}\,\text{s}^{-1}$$ **1.0**

The new velocity will be given by (we use $'$ symbol to denote quantities after the first orbit-raising maneuvre)

$$v_p' = v_p + \Delta v = 10.06\,\text{km}\,\text{s}^{-1}$$ **1.0**

The perigee remains unchanged. So we get $r_p' = r_p$. **1.0**
Since the satellite is moving faster, the new apogee will be higher.

$$v_p' = \sqrt{2GM_\oplus \frac{r_a'}{r_p'(r_a' + r_p')}}$$

$$\therefore 1 + \frac{r_p'}{r_a'} = \frac{2GM_\oplus}{(v_p')^2 \times r_p'}$$ **2.0**

$$= \frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{(10.06 \times 10^3)^2 \times 6.635 \times 10^6} = 1.188$$

$$r_a' = \frac{6635}{0.188} = 35\,380\,\text{km}$$ **2.0**

$$h_a = 35380 - 6371$$

$$= \boxed{29\,009\,\text{km}}$$ **1.0**

Acceptable range: $\pm 150\,\mathrm{km}$

(T9.2) Find eccentricity (e) of the new orbit after the burn and new orbital period (P) of MOM in hours. **6**

Solution:
As seen above,

$$\frac{r'_\mathrm{p}}{r'_\mathrm{a}} = 0.188 = \frac{1-e}{1+e} \quad \mathbf{1.0}$$

$$\therefore e = \frac{1-0.188}{1+0.188} = \boxed{0.683} \quad \mathbf{1.0}$$

Acceptable range: ± 0.002
The new orbital semi-major axis and orbital period will be,

$$a' = \frac{r'_\mathrm{a} + r'_\mathrm{p}}{2} \quad \mathbf{1.0}$$

$$= \frac{35380 + 6635}{2} = 20\,933\,\mathrm{km} \quad \mathbf{1.0}$$

$$P = 2\pi\sqrt{\frac{a'^3}{GM_\oplus}} \quad \mathbf{1.0}$$

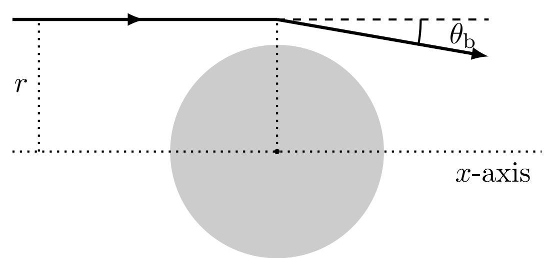
$$= 2\pi\sqrt{\frac{(2.0933\times 10^7)^3}{6.674\times 10^{-11}\times 5.972\times 10^{24}}} = \boxed{30\,136\,\mathrm{s} = 8.37\,\mathrm{h}} \quad \mathbf{1.0}$$

Acceptable range: $\pm 0.1\,\mathrm{h}$

(T10) **Gravitational Lensing Telescope**
Einstein's General Theory of Relativity predicts bending of light around massive bodies. For simplicity, we assume that the bending of light happens at a single point for each light ray, as shown in the figure. The angle of bending, θ_b, is given by

$$\theta_\mathrm{b} = \frac{2R_\mathrm{sch}}{r}$$

where R_sch is the Schwarzschild radius associated with that gravitational body. We call r, the distance of the incoming light ray from the parallel x-axis passing through the centre of the body, as the "impact parameter".

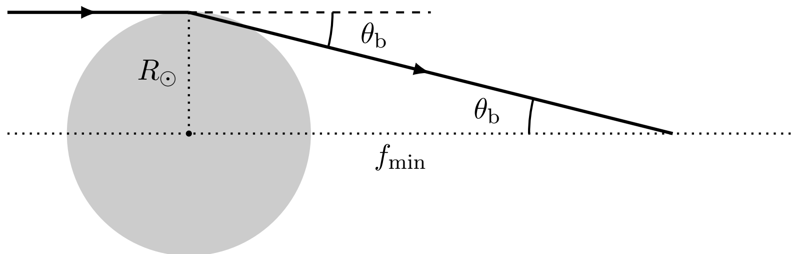


A massive body thus behaves somewhat like a focusing lens. The light rays coming from infinite distance beyond a massive body, and having the same impact parameter r, converge at a point along the axis, at a distance f_r from the centre of the massive body. An observer at that point will benefit from huge amplification due to this gravitational focusing. The massive body in this case is being used as a Gravitational Lensing Telescope for amplification of distant signals.

(T10.1) Consider the possibility of our Sun as a gravitational lensing telescope. Calculate the shortest distance, $f_{\rm min}$, from the centre of the Sun (in A.U.) at which the light rays can get focused. **6**

Solution:

The rays travelling closer to the gravitational body will bend more. Thus, we get shortest convergence point where the rays just grazing the solar surface will meet each other.



$$\theta_{\rm b} = \frac{2R_{\rm sch}}{R_\odot} \simeq \frac{R_\odot}{f_{\rm min}}$$ **2.0**

$$\therefore f_{\rm min} = \frac{R_\odot^2}{2R_{\rm sch}} = \frac{R_\odot^2 c^2}{4GM_\odot}$$ **2.0**

$$= \frac{(6.955\times10^8 \times 2.998\times10^8)^2}{4\times6.674\times10^{-11}\times1.989\times10^{30}}\,\text{m}$$

$$= 8.188\times10^{13}\,\text{m} = \frac{8.188\times10^{13}}{1.496\times10^{11}}\,\text{AU}$$

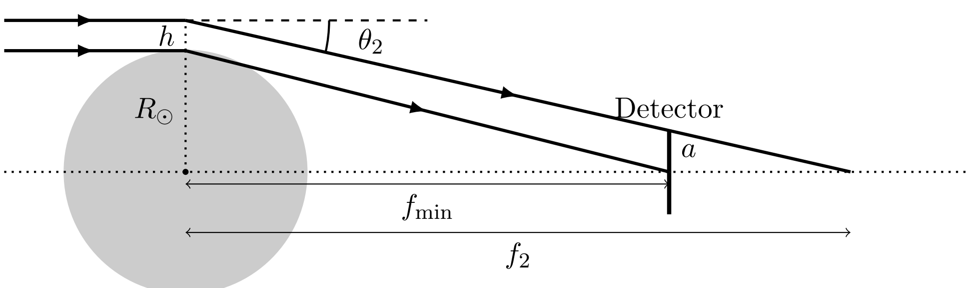
$$\boxed{f_{\rm min} = 547.3\,\text{AU}}$$ **2.0**

(T10.2) Consider a small circular detector of radius a, kept at a distance $f_{\rm min}$ centred on the x-axis and perpendicular to it. Note that only the light rays which pass within a certain annulus (ring) of width h (where $h \ll R_\odot$) around the Sun would encounter the detector. The amplification factor at the detector is defined as the ratio of the intensity of the light incident on the detector in the presence of the Sun and the intensity in the absence of the Sun. **8**

Express the amplification factor, $A_{\rm m}$, at the detector in terms of $R_\odot$ and a.

Solution:

The following figure needs to be drawn. **1.0**



The light bending from the surface of the Sun ($r = R_\odot$) will intersect the detector at its centre, as it is kept at $f_{\rm min}$.
The boundary of the detector will be intersected by a light ray with $r = R_\odot + h$. This ray will intersect the x-axis at a distance f_2.

$$f_2 = \frac{(R_\odot + h)^2}{2R_{\rm sch}}$$ **2.0**

Same argument as in Part 1
For small angles,

$$\begin{aligned} a &= (f_2 - f_{\rm min})\theta_2 \\ &= \left[\frac{(R_\odot + h)^2}{2R_{\rm sch}} - \frac{R_\odot^2}{2R_{\rm sch}}\right] \frac{2R_{\rm sch}}{(R_\odot + h)} = \frac{2R_\odot h + h^2}{R_\odot + h} \\ &\simeq 2h \end{aligned}$$ **2.0**

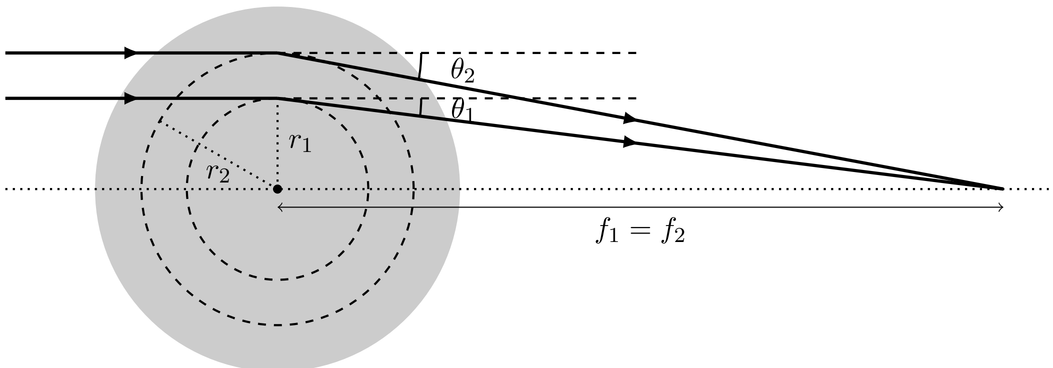
Let original intensity of the incoming radiation be I_0.
The flux at the detector in the presence of Sun is $\Phi_\odot = I_0\ 2\pi\ R_\odot\ h$ **1.0**
The flux at the detector in the absence of Sun is $\Phi_0 = I_0\ \pi\ a^2$ **1.0**
The amplification is therefore

$$A_{\rm m} = \frac{\Phi_\odot}{\Phi_0} = \frac{I_0 2\pi R_\odot h}{I_0 \pi a^2} = \boxed{\frac{R_\odot}{a}}$$ **1.0**

(T10.3) Consider a spherical mass distribution, such as a dark matter cluster, through which light rays can pass while undergoing gravitational bending. Assume for simplicity that for the gravitational bending with impact parameter, r, only the mass $M(r)$ enclosed inside the radius r is relevant. **6**

What should be the mass distribution, $M(r)$, such that the gravitational lens behaves like an ideal optical convex lens ?

Solution:



All rays should focus at the same spot. This should be evident from figure drawn on answersheet or otherwise. **2.0**

Let there be two rays with impact parameters r_1 and r_2. The corresponding distances of focus will be

$$f_i = \frac{r_i^2}{2r_{\rm sch_i}} = \frac{r_i^2 c^2}{4GM(r_i)}$$ **2.0**

Same argument as in Part 1

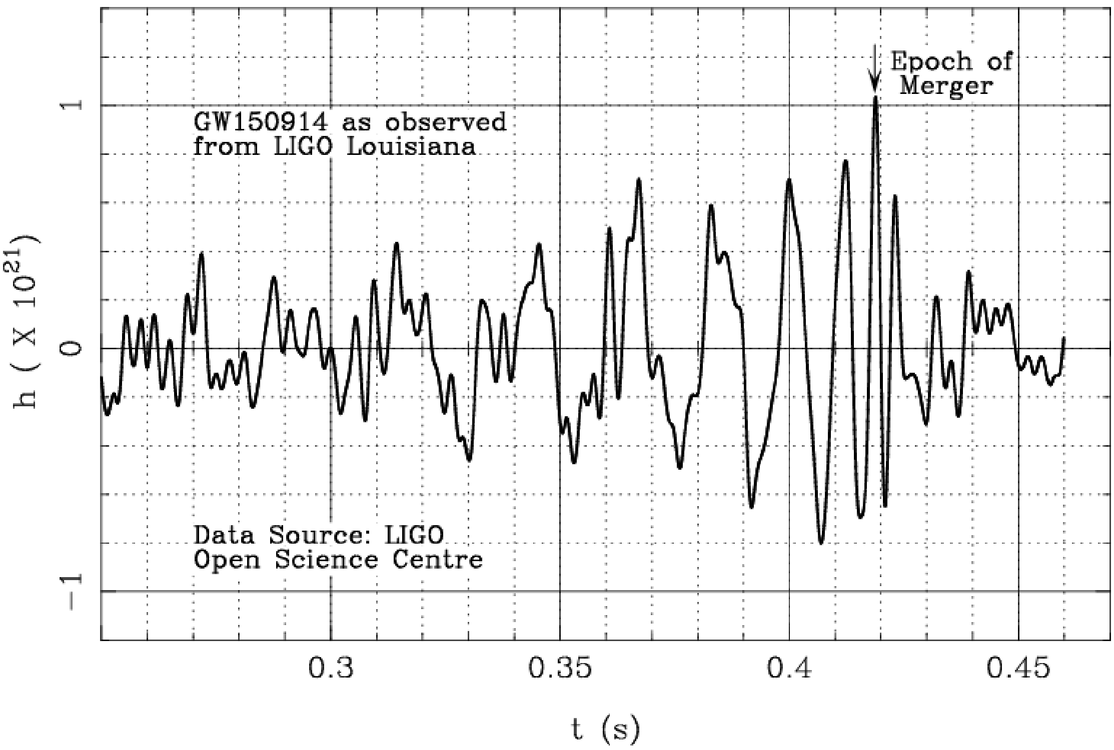
The rquirement $f_1 = f_2$ implies

$$\frac{r_1^2}{r_2^2} = \frac{M(r_1)}{M(r_2)}$$ **1.0**

The required mass distribution is: $\boxed{M(r) \propto r^2}$ **1.0**

(T11) **Gravitational Waves**

The first signal of gravitational waves was observed by two advanced LIGO detectors at Hanford and Livingston, USA in September 2015. One of these measurements (strain vs time in seconds) is shown in the accompanying figure. In this problem, we will interpret this signal in terms of a small test mass m orbiting around a large mass M (i.e., $m \ll M$), by considering several models for the nature of the central mass.



The test mass loses energy due to the emission of gravitational waves. As a result the orbit keeps on shrinking, until the test mass reaches the surface of the object, or in the case of a black hole, the innermost stable circular orbit – ISCO – which is given by $R_{\text{ISCO}} = 3R_{\text{sch}}$, where R_{sch} is the Schwarzschild radius of the black hole. This is the "epoch of merger". At this point, the amplitude of the gravitational wave is maximum, and so is its frequency, which is always twice the orbital frequency. In this problem, we will only focus on the gravitational waves before the merger, when Kepler's laws are assumed to be valid. After the merger, the form of gravitational waves will drastically change.

(T11.1) Consider the observed gravitational waves shown in the figure above. Estimate the time period, T_0, and hence calculate the frequency, f_0, of gravitational waves just before the epoch of merger. **3**

Solution:
From the graph, just before the peak of emission, the time period of gravitational waves is approximately $(0.007 \pm 0.004)\,\mathrm{s}$.

$$T_0 \approx \boxed{0.007\,\mathrm{s}}$$ **2.0**

Acceptable range: 0.003 to 0.011 s
That is, the frequency of the gravitational waves is $f_0 \approx \boxed{142.86\,\mathrm{Hz}}$. **1.0**
Acceptable range: 333.33 to 90.91 Hz
Answer given in terms of angular frequency with correct value and units gains full credit.

(T11.2) For any main sequence (MS) star, the radius of the star, R_{MS}, and its mass, M_{MS}, are related by a power law given as, **10**

$$R_{\mathrm{MS}} \propto (M_{\mathrm{MS}})^{\alpha}$$
$$\text{where } \alpha = 0.8 \quad \text{for } M_\odot < M_{\mathrm{MS}}$$
$$= 1.0 \quad \text{for } 0.08M_\odot \leq M_{\mathrm{MS}} \leq M_\odot$$

If the central object were a main sequence star, write an expression for the maximum frequency of gravitational waves, f_{MS}, in terms of mass of the star in units of solar masses $(M_{\mathrm{MS}}/M_\odot)$ and α.

Solution:
Since $m \ll M$ then, by Kepler's third law

$$f_{\mathrm{orbital}} = \frac{1}{2\pi}\sqrt{\frac{GM}{r^3}}$$ **4.0**

Hence the frequency of the gravitational waves is

$$f_{\mathrm{grav}} = 2f_{\mathrm{orbital}} = \frac{1}{\pi}\sqrt{\frac{GM}{r^3}}$$ **1.0**

The frequency will be maximum when $r = R_{\mathrm{MS}}$. **1.0**

For main sequence stars,

$$\frac{R_{\mathrm{MS}}}{R_\odot} = \left(\frac{M_{\mathrm{MS}}}{M_\odot}\right)^{\alpha}$$

$$\therefore R_{\mathrm{MS}} = R_\odot\left(\frac{M_{\mathrm{MS}}}{M_\odot}\right)^{\alpha}$$ **1.0**

$$\therefore f_{\mathrm{MS}} = \frac{1}{\pi}\sqrt{\frac{GM_{\mathrm{MS}}}{R_\odot^3}}\left(\frac{M_\odot}{M_{\mathrm{MS}}}\right)^{3\alpha/2}$$

$$= \frac{1}{\pi}\sqrt{\frac{GM_\odot}{R_\odot^3}}\left(\frac{M_\odot}{M_{\mathrm{MS}}}\right)^{(3\alpha-1)/2}$$

$$\boxed{f_{\mathrm{MS}} = \frac{1}{\pi}\sqrt{\frac{GM_\odot}{R_\odot^3}}\left(\frac{M_{\mathrm{MS}}}{M_\odot}\right)^{(1-3\alpha)/2}}$$ **3.0**

(T11.3) Using the above result, determine the appropriate value of α that will give the maximum possible frequency of gravitational waves, $f_{\mathrm{MS,max}}$ for any main sequence star. Evaluate **9** this frequency.

Solution:
For possible values of α given in the question, the exponent $\frac{1-3\alpha}{2}$ is negative. Thus, if $M_{\rm MS} > M_\odot$, the frequency will be smaller. Thus, for highest possible frequency coming from a main sequence star, you should take lowest possible mass i.e. $\boxed{\alpha = 1.0}$ **4.0**

$$f_{\rm MS,max} = \frac{1}{\pi}\sqrt{\frac{GM_\odot}{R_\odot^3}\left(\frac{M_{\rm MS}}{M_\odot}\right)^{\left(\frac{1-3\times1}{2}\right)}}$$
$$= \frac{1}{\pi}\sqrt{\frac{GM_\odot}{R_\odot^3}\times\frac{M_\odot}{M_{\rm MS}}}$$ **2.0**

The frequency of gravitational waves will be given by,

$$f_{\rm MS,max} = \frac{1}{\pi}\sqrt{\frac{6.674\times10^{-11}\times1.989\times10^{30}}{(6.955\times10^8)^3}\times\frac{1}{0.08}}$$
$$\boxed{f_{\rm MS,max} = 2.5\,\text{mHz}}$$ **3.0**

Answer given in terms of angular frequency with correct value and units gains full credit.

(T11.4) White dwarf (WD) stars have a maximum mass of $1.44M_\odot$ (known as the Chandrasekhar limit) and obey the mass-radius relation $R \propto M^{-1/3}$. The radius of a solar mass white dwarf is equal to 6000 km. Find the highest frequency of emitted gravitational waves, $f_{\rm WD,max}$, if the test mass is orbiting a white dwarf. **8**

Solution:
The maximum frequency would be when $r = R_{\rm WD}$. **1.0**
We use the notation $R_{WD\odot}$ for the radius of a solar mass white dwarf. Then for white dwarfs

$$R_{WD}^3 = R_{WD\odot}^3\frac{M_{WD}}{M_\odot}$$ **1.0**

$$f_{\rm WD} = \frac{1}{\pi}\sqrt{\frac{GM_{\rm WD}}{R_{WD}^3}}$$
$$= \frac{1}{\pi}\sqrt{\frac{GM_\odot}{R_{WD\odot}^3}\frac{M_{WD}}{M_\odot}}$$ **2.0**

For maximum frequency, we have to take highest white dwarf mass. **2.0**

$$f_{\rm WD,max} = 2.600\times10^{-6}\times\sqrt{\frac{1.989\times10^{30}}{(6000\times10^3)^3}\times1.44}$$
$$= 2.600\times10^{-6}\times95.96\times10^3\times1.44$$
$$\boxed{f_{\rm WD,max} = 0.359\,\text{Hz}}$$ **2.0**

Answer given in terms of angular frequency with correct value and units gains full credit.

(T11.5) Neutron stars (NS) are a peculiar type of compact objects which have masses between **8**

1 and $3M_\odot$ and radii in the range 10 – 15 km. Find the range of frequencies of emitted gravitational waves, $f_{\mathrm{NS,min}}$ and $f_{\mathrm{NS,max}}$, if the test mass is orbiting a neutron star at a distance close to the neutron star radius.

Solution:

$$f_{grav} = \frac{1}{\pi}\sqrt{\frac{GM_{\mathrm{NS}}}{R_{\mathrm{NS}}^3}}$$

The lowest possible frequency is when M_{NS} is lowest and R_{NS} is the highest. **2.0**

For $M_{\mathrm{NS}} = M_\odot$ and $R = 15\,\mathrm{km}$, we get

$\boxed{f_{\mathrm{NS,min}} = 1.996\,\mathrm{kHz}}$ **2.0**

Similarly, the largest possible frequency is when M_{NS} is the largest and R_{NS} is the smallest. **2.0**

For $M_{\mathrm{NS}} = 3M_\odot$ and $R = 10\,\mathrm{km}$, we get

$\boxed{f_{\mathrm{NS,max}} = 6.352\,\mathrm{kHz}}$ **2.0**

Answer given in terms of angular frequency with correct value and units gains full credit.

(T11.6) If the test mass is orbiting a black hole (BH), write the expression for the frequency of emitted gravitational waves, f_{BH}, in terms of mass of the black hole, M_{BH}, and the solar mass $M_\odot$. **7**

Solution:

For black holes, we have to consider R_{ISCO}. **2.0**

Hence the equation will be,

$$f_{BH} = \frac{1}{\pi} \times \sqrt{\frac{GM_\odot}{27R^3_{sch-\odot}}} \times \frac{M_\odot}{M_{BH}}$$ **2.0**

$$\boxed{f_{BH} = 4.396\,\mathrm{kHz} \times \frac{M_\odot}{M_{BH}}}$$ **3.0**

Answer given in terms of angular frequency with correct value and units gains full credit.

(T11.7) Based only on the time period (or frequency) of gravitational waves before the epoch of merger, determine whether the central object can be a main sequence star (MS), a white dwarf (WD), a neutron star (NS), or a black hole (BH). Tick the correct option in the Summary Answersheet. Estimate the mass of this object, M_{obj}, in units of $M_\odot$. **5**

Solution:

We found the frequency of the LIGO-detected wave to be 166.67 Hz just before merger. As per our analysis above, only black holes can lead to emission in this frequency range.

$\boxed{\text{Black Hole}}$ **2.0**

By using corresponding expression,

$$\boxed{M_{\mathrm{obj}} = \frac{4396}{142.86} M_\odot \approx 31 M_\odot}$$ **3.0**

Any answer between 13 to 50 will get full credit.

(T12) **Exoplanets**

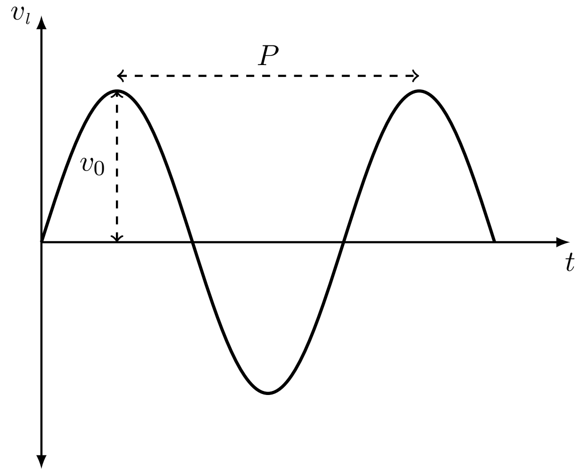
Two major methods of detection of exoplanets (planets around stars other than the Sun) are the radial velocity (or so-called "wobble") method and the transit method. In this problem, we find out how a combination of the results of these two methods can reveal a lot of information about an orbiting exoplanet and its host star.

Throughout this problem, we consider the case of a planet of mass M_p and radius R_p moving in a circular orbit of radius a around a star of mass M_s ($M_\text{s} \gg M_\text{p}$) and radius R_s. The normal to the orbital plane of the planet is inclined at angle i with respect to the line of sight ($i = 90^\circ$ would mean "edge on" orbit). We assume that there is no other planet orbiting the star and $R_\text{s} \ll a$.

"Wobble" Method:

When a planet and a star orbit each other around their barycentre, the star is seen to move slightly, or "wobble", since the centre of mass of the star is not coincident with the barycentre of the star-planet system. As a result, the light received from the star undergoes a small Doppler shift related to the velocity of this wobble.

The line of sight velocity, v_l, of the star can be determined from the Doppler shift of a known spectral line, and its periodic variation with time, t, is shown in the schematic diagram below. In the diagram, the two measurable quantities in this method, namely, the orbital period P and maximum line of sight velocity v_0 are shown.



(T12.1) Derive expressions for the orbital radius (a) and orbital speed (v_p) of the planet in terms of M_s and P. **3**

Solution:

Kepler's law:

$$\boxed{a = \left(\frac{GM_\text{s}}{4\pi^2}P^2\right)^{1/3}} \qquad \textbf{1.0}$$

Gravitational force provides centripetal acceleration:

$$v_\text{p} = \sqrt{\frac{GM_\text{s}}{a}} \qquad \textbf{0.5}$$

$$\boxed{v_\text{p} = \left(\frac{2\pi GM_\text{s}}{P}\right)^{1/3}} \qquad \textbf{1.5}$$

(T12.2) Obtain a lower limit on the mass of the planet, $M_{\mathrm{p,\,min}}$ in terms of M_s, v_0 and v_p. **4**

Solution:
Momentum conservation:

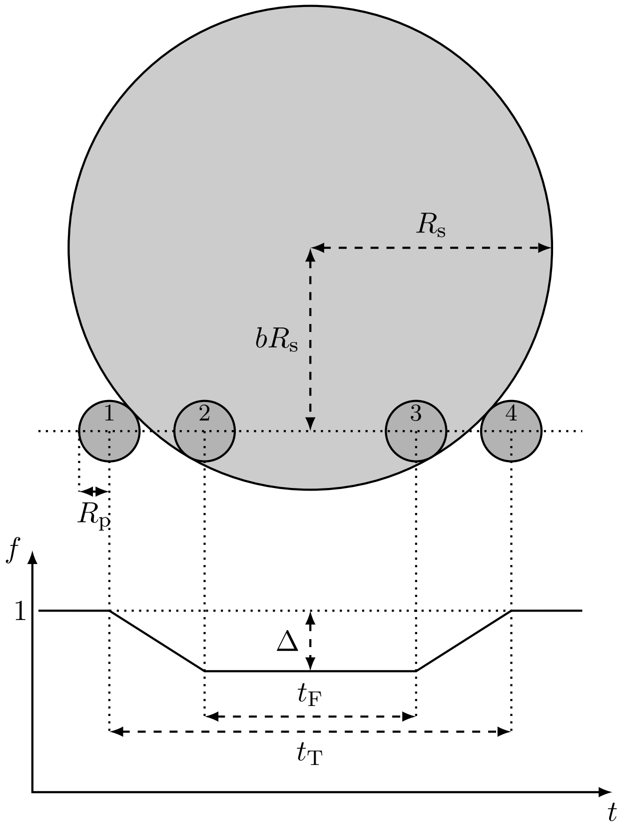
$$M_\mathrm{p} v_\mathrm{p} = M_\mathrm{s} v_\mathrm{s}$$ **1.5**

Observed quantity is $v_0 = v_\mathrm{s} \sin i$. Thus,

$$M_{\mathrm{p,\,min}} = M_\mathrm{p} \sin i = \frac{M_\mathrm{s} v_\mathrm{s} \sin i}{v_\mathrm{p}} = \boxed{\frac{M_\mathrm{s} v_0}{v_\mathrm{p}}}$$

This is a lower limit on M_p. **2.5**

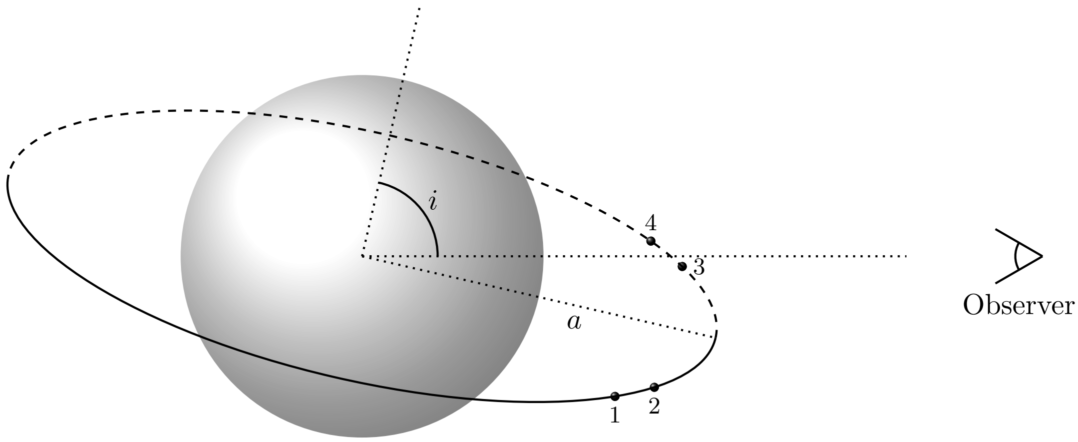
Transit Method:
As a planet orbits its host star, for orientations of the orbital plane that are close to "edge-on" ($i \approx 90°$), it will pass periodically, or "transit", in front of the stellar disc as seen by the observer. This would cause a tiny decrease in the observed stellar flux which can be measured. The schematic diagram below (NOT drawn to scale) shows the situation from the observer's perspective and the resulting transit light curve (normalised flux, f, vs time, t) for a uniformly bright stellar disc.



If the inclination angle i is exactly $90°$, the planet would be seen to cross the stellar disc along a diameter. For other values of i, the transit occurs along a chord, whose centre lies at a distance bR_s from the centre of the stellar disc, as shown. The no-transit flux is normalised to 1 and the maximum dip during the transit is given by Δ.

The four significant points in the transit are the first, second, third and fourth contacts, marked by the positions 1 to 4, respectively, in the figure above. The time interval during the second and third contacts is denoted by t_{F}, when the disc of the planet overlaps the stellar disc fully. The time interval between the first and fourth contacts is denoted by t_{T}.

These points are also marked in the schematic diagram below showing a "side-on" view of the orbit (NOT drawn to scale).



The measurable quantities in the transit method are P, t_{T}, t_{F} and Δ.

(T12.3) Find the constraint on i in terms of R_{s} and a for the transit to be visible at all to the distant observer. **2**

Solution:

$$bR_{\mathrm{s}} = a\cos i$$ **1.0**

Therefore, for visibility, $0 \leq b \leq 1 \Rightarrow i \geq \cos^{-1}(R_{\mathrm{s}}/a)$ **1.0**

(T12.4) Express Δ in terms of R_{s} and R_{p}. **1**

Solution:

Blackbody $\Rightarrow$ brightness is proportional to area. Since the observer is far away from the star-planet system, size of silhouette of planet on stellar disc is independent of a.

$$\Delta = \left(\frac{R_{\mathrm{p}}}{R_{\mathrm{s}}}\right)^2$$ **1.0**

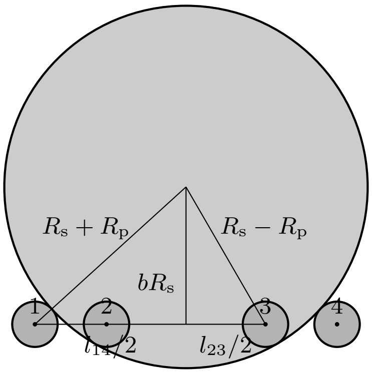
(T12.5) Express t_{T} and t_{F} in terms of R_{s}, R_{p}, a, P and b. **8**

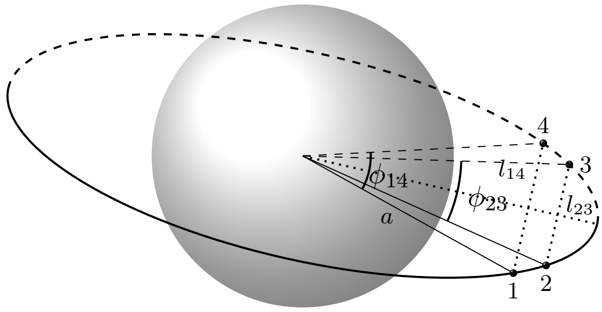
Solution:

Circular orbit $\Rightarrow$ uniform orbital speed

$$\Rightarrow \frac{t}{P} = \frac{a\phi}{2\pi a} = \frac{\phi}{2\pi}$$ **1.0**

where ϕ is the angle subtended by the planet at the centre of the star during transit (over time t).





$$(l_{23}/2)^2 = (R_s - R_p)^2 - (bR_s)^2$$

$$l_{23} = 2R_s\sqrt{(1 - R_p/R_s)^2 - b^2}$$ **2.0**

$$\sin(\phi_{23}/2) = \frac{l_{23}/2}{a}$$ **2.0**

$$\Rightarrow \phi_{23} = 2\sin^{-1}\left(\frac{l_{23}}{2a}\right) = 2\sin^{-1}\left[\frac{R_s}{a}\sqrt{\left(1 - \frac{R_p}{R_s}\right)^2 - b^2}\right]$$ **1.0**

$$t_F = \frac{P}{2\pi}\phi_{23} = \frac{P}{\pi}\sin^{-1}\left[\frac{R_s}{a}\sqrt{\left(1 - \frac{R_p}{R_s}\right)^2 - b^2}\right]$$ **1.0**

Similarly,

$$t_T = \frac{P}{\pi}\sin^{-1}\left[\frac{R_s}{a}\sqrt{\left(1 + \frac{R_p}{R_s}\right)^2 - b^2}\right]$$ **1.0**

(T12.6) In the approximation of an orbit much larger than the stellar radius, show that the parameter b is given by **5**

$$b = \left[1 + \Delta - 2\sqrt{\Delta}\frac{1 + \left(\frac{t_F}{t_T}\right)^2}{1 - \left(\frac{t_F}{t_T}\right)^2}\right]^{1/2}$$

Solution:
Since $R_s \ll a$, use $\sin^{-1} x \approx x$. **2.0**

$$t_T \approx \frac{P}{\pi}\left[\frac{R_s}{a}\sqrt{\left(1 + \frac{R_p}{R_s}\right)^2 - b^2}\right]$$

$$t_F \approx \frac{P}{\pi}\left[\frac{R_s}{a}\sqrt{\left(1 - \frac{R_p}{R_s}\right)^2 - b^2}\right]$$

Dividing, and putting $R_p/R_s = \sqrt{\Delta}$,

$$\frac{t_F}{t_T} = \left[\frac{(1-\sqrt{\Delta})^2 - b^2}{(1+\sqrt{\Delta})^2 - b^2}\right]^{1/2}$$ **1.0**

$$\Rightarrow b = \left[\frac{(1-\sqrt{\Delta})^2 - \left(\frac{t_F}{t_T}\right)^2 (1+\sqrt{\Delta})^2}{1-\left(\frac{t_F}{t_T}\right)^2}\right]^{1/2} = \left[1+\Delta-2\sqrt{\Delta}\frac{1+\left(\frac{t_F}{t_T}\right)^2}{1-\left(\frac{t_F}{t_T}\right)^2}\right]^{1/2}$$ **2.0**

Expressions lacking the use of approximation $R_s/a \ll 1$, but otherwise correct will get a penalty of 2.0.
Use of approximation with proper justification at a later stage than at the first step will get full credit.

(T12.7) Use the result of part (T12.6) to obtain an expression for the ratio a/R_s in terms of measurable transit parameters, using a suitable approximation. **3**

Solution:

$$t_T = \frac{P}{\pi}\left[\frac{R_s}{a}\sqrt{\left(1+\frac{R_p}{R_s}\right)^2 - b^2}\right]$$ **1.0**

Either substitution of b or elimination of b gets 1.0.

Substituting b and R_p/R_s,

$$t_T = \frac{P}{\pi}\left[\frac{R_s}{a}\sqrt{(1+\sqrt{\Delta})^2 - 1 - \Delta + 2\sqrt{\Delta}\frac{1+\left(\frac{t_F}{t_T}\right)^2}{1-\left(\frac{t_F}{t_T}\right)^2}}\right]$$

$$\Rightarrow t_T = \frac{P}{\pi}\frac{R_s}{a}\left[\frac{4\sqrt{\Delta}}{1-\left(\frac{t_F}{t_T}\right)^2}\right]^{1/2}$$

$$\Rightarrow \frac{a}{R_s} = \frac{2P\Delta^{1/4}}{\pi(t_T^2 - t_F^2)^{1/2}}$$ **2.0**

Expressions lacking the use of approximation $R_s/a \ll 1$, but otherwise correct will get a penalty of 1.0.
If penalty has already been imposed in part (T12.6), no further penalty for lack of approximation.

(T12.8) Combine the results of the wobble method and the transit method to determine the stellar mean density $\rho_s \equiv \frac{M_s}{4\pi R_s^3/3}$ in terms of t_T, t_F, Δ and P. **6**

Solution:
From part (T12.7)

$$a = R_s \frac{2P\Delta^{1/4}}{\pi(t_T^2 - t_F^2)^{1/2}}$$

From part (T12.1)

$$a = \left(\frac{GM_s}{4\pi^2}P^2\right)^{1/3}$$

Combining,

$$\frac{GM_s}{4\pi^2}P^2 = \left[R_s \frac{2P\Delta^{1/4}}{\pi(t_T^2 - t_F^2)^{1/2}}\right]^3$$ **3.0**

Identifying the two equations for combining gets credit. No credit for writing only one equation or irrelevant equations.

$$\Rightarrow \rho_s \equiv \frac{M_s}{4\pi R_s^3/3} = \frac{3}{4\pi}\frac{4\pi^2}{P^2 G}\frac{8P^3\Delta^{3/4}}{\pi^3(t_T^2 - t_F^2)^{3/2}}$$ **2.0**

$$\Rightarrow \boxed{\rho_s = \frac{24}{\pi^2 G}\frac{P(\Delta)^{3/4}}{(t_T^2 - t_F^2)^{3/2}}}$$ **1.0**

Rocky or gaseous:
Let us consider an edge-on ($i = 90°$) star-planet system (circular orbit for the planet), as seen from the Earth. It is known that the host star is of mass $1.00 M_\odot$. Transits are observed with a period (P) of 50.0 days and total transit duration (t_T) of 1.00 hour. The transit depth (Δ) is 0.0064. The same system is also observed in the wobble method to have a maximum line of sight velocity of $0.400\,\mathrm{m\,s^{-1}}$.

(T12.9) Find the orbital radius a of the planet in units of AU and in metres. **2**

Solution:
From Kepler's third law (with same mass of host star):

$$\frac{a}{a_\oplus} = \left(\frac{P}{P_\oplus}\right)^{2/3}$$ **1.0**

$$a = \left(\frac{50.0}{365.242}\right)^{2/3} \times 1\,\mathrm{AU} = \boxed{0.266\,\mathrm{AU}}$$ **0.5**

$$= 0.266 \times 1.496 \times 10^{11}\,\mathrm{m} = \boxed{3.97 \times 10^{10}\,\mathrm{m}}$$ **0.5**

(T12.10) Find the ratio t_F/t_T of the system. **2**

Solution:
Edge-on $\Rightarrow b = 0$ **1.0**

$$\frac{t_F}{t_T} = \left[\frac{(1-\sqrt{\Delta})^2 - b^2}{(1+\sqrt{\Delta})^2 - b^2}\right]^{1/2} = \frac{1-\sqrt{\Delta}}{1+\sqrt{\Delta}} = \boxed{0.8519}$$ **1.0**

(T12.11) Obtain the mass M_p and radius R_p of the planet in terms of the mass ($M_\oplus$) and radius ($R_\oplus$) of the Earth respectively. Is the composition of the planet likely to be rocky or gaseous? Tick the box for ROCKY or GASEOUS in the Summary Answersheet. **8**

Solution:

From parts (T12.1) and (T12.9)

$$v_\text{p} = \sqrt{\frac{GM_\odot}{a}} = \sqrt{\frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{3.97 \times 10^{10}}} = 57.798\,\text{km}\,\text{s}^{-1}$$ **1.0**

Assumption of small planet ($M_\text{p} \ll M_\text{s}$) is valid because Δ is very small; less dense planet would make the assumption stronger!

$$M_\text{p} = \frac{M_\text{s} v_0}{v_\text{p}} = \frac{1.989 \times 10^{30} \times 0.400}{5.7798 \times 10^4 \times 5.972 \times 10^{24}} M_\oplus = \boxed{2.30\, M_\oplus}$$ **2.0**

From part (T12.4),

$$R_\text{p} = R_\text{s}\sqrt{\Delta}$$

From part (T12.7),

$$\frac{a}{R_\text{s}} = \frac{2P\Delta^{1/4}}{\pi(t_\text{T}^2 - t_\text{F}^2)^{1/2}} = \frac{2P\Delta^{1/4}}{\pi t_\text{T}(1-(t_\text{F}/t_\text{T})^2)^{1/2}}$$

$$\therefore R_\text{s} = \frac{a\pi t_\text{T}(1-(t_\text{F}/t_\text{T})^2)^{1/2}}{2P\Delta^{1/4}}$$

Combining,

$$\begin{aligned} R_\text{p} &= \frac{a\pi t_\text{T}(1-(t_\text{F}/t_\text{T})^2)^{1/2}\Delta^{1/2}}{2P\Delta^{1/4}} \\ &= \frac{a\pi t_\text{T}(1-(t_\text{F}/t_\text{T})^2)^{1/2}\Delta^{1/4}}{2P} \\ &= \frac{3.97 \times 10^{10} \times \pi \times \frac{1}{24} \times (1-0.8519^2)^{1/2} \times (0.0064)^{1/4}}{2 \times 50.0 \times 6.371 \times 10^6} R_\oplus \\ &= \boxed{1.21\, R_\oplus} \end{aligned}$$ **2.0** **1.0**

Mean density

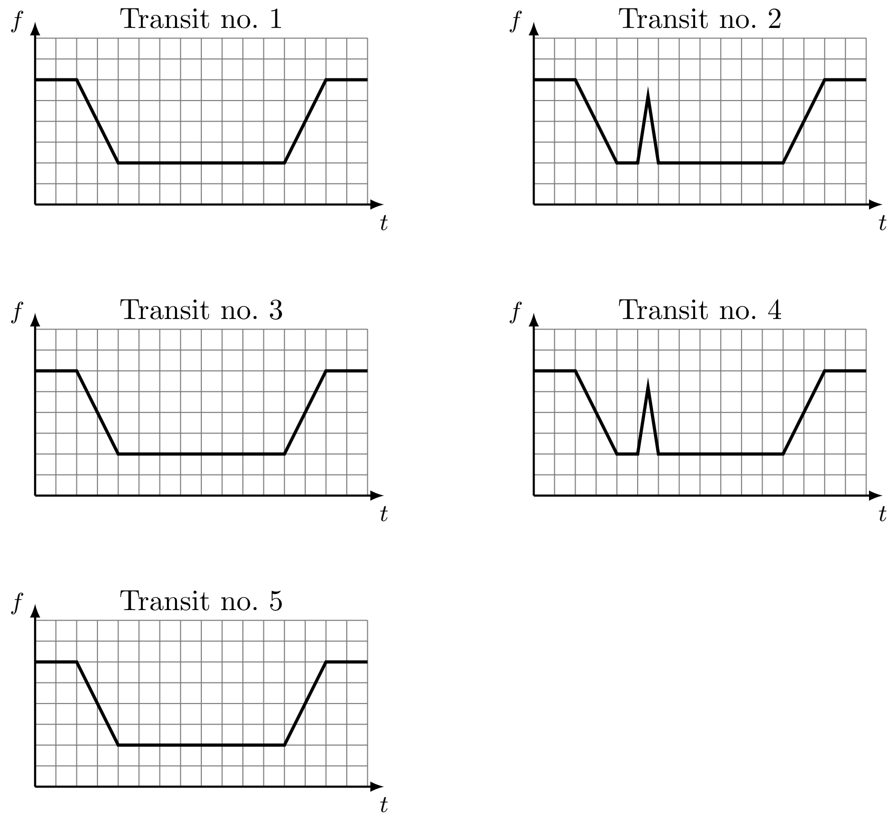
$$\rho_\text{p} = \frac{M_\text{p}}{4\pi R_\text{p}^3/3} = \frac{2.30}{(1.21)^3}\rho_\oplus = 1.3\rho_\oplus$$ **1.0**

Since mean density is higher than that of Earth, the planet is $\boxed{\text{Rocky}}$. **1.0**

Transit light curves with starspots and limb darkening:

(T12.12) Consider a planetary transit with $i = 90°$ around a star which has a starspot on its equator, comparable to the size of the planet, R_p. The rotation period of the star is $2P$. Draw schematic diagrams of the transit light curve for five successive transits of the planet (in the templates provided in the Summary Answersheet). The no-transit flux for each transit may be normalised to unity independently. Assume that the planet does not encounter the starspot on the first transit but does in the second. **4**

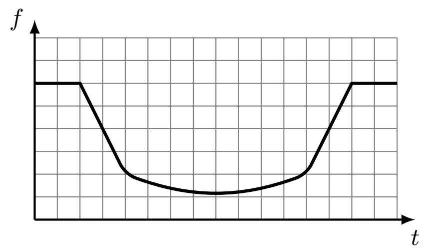
Solution:



- **No change in the first: 0.5**
- **Spike in second (width and phase of spike is arbitrary): 1.0**
- **Height of spike (almost) equal to maximum dip: 0.5**
- **No change in third: 0.5**
- **Spike again in fourth: 0.5**
- **Same phase of spike in second and fourth: 0.5**
- **No change in 5th: 0.5**

(T12.13) Throughout the problem we have considered a uniformly bright stellar disc. However, real stellar discs have limb darkening. Draw a schematic transit light curve when limb darkening is present in the host star. **2**

Solution:



Non-flat bottom with central minimum gets 2.0. Curvature of ingress and egress are tolerated.

(D1) **Binary Pulsar**

Through systematic searches during the past decades, astronomers have found a large number of millisecond pulsars (spin period $< 10\,\mathrm{ms}$). Majority of these pulsars are found in binaries, with nearly circular orbits.

For a pulsar in a binary orbit, the measured pulsar spin period (P) and the measured line-of-sight acceleration (a) both vary systematically due to orbital motion. For circular orbits, this variation can be described mathematically in terms of orbital phase ϕ ($0 \le \phi \le 2\pi$) as,

$$P(\phi) = P_0 + P_\mathrm{t}\cos\phi \qquad \text{where } P_\mathrm{t} = \frac{2\pi P_0 r}{cP_\mathrm{B}}$$

$$a(\phi) = -a_\mathrm{t}\sin\phi \qquad \text{where } a_\mathrm{t} = \frac{4\pi^2 r}{P_\mathrm{B}^2}$$

where P_B is the orbital period of the binary, P_0 is the intrinsic spin period of the pulsar and r is the radius of the orbit.

The following table gives one such set of measurements of P and a at different heliocentric epochs, T, expressed in truncated Modified Julian Days (tMJD), i.e. number of days since MJD = 2,440,000.

No.	T (tMJD)	P (µs)	a ($\mathrm{m\,s^{-2}}$)
1	5740.654	7587.8889	-0.92 ± 0.08
2	5740.703	7587.8334	-0.24 ± 0.08
3	5746.100	7588.4100	-1.68 ± 0.04
4	5746.675	7588.5810	$+1.67 \pm 0.06$
5	5981.811	7587.8836	$+0.72 \pm 0.06$
6	5983.932	7587.8552	-0.44 ± 0.08
7	6005.893	7589.1029	$+0.52 \pm 0.08$
8	6040.857	7589.1350	$+0.00 \pm 0.04$
9	6335.904	7589.1358	$+0.00 \pm 0.02$

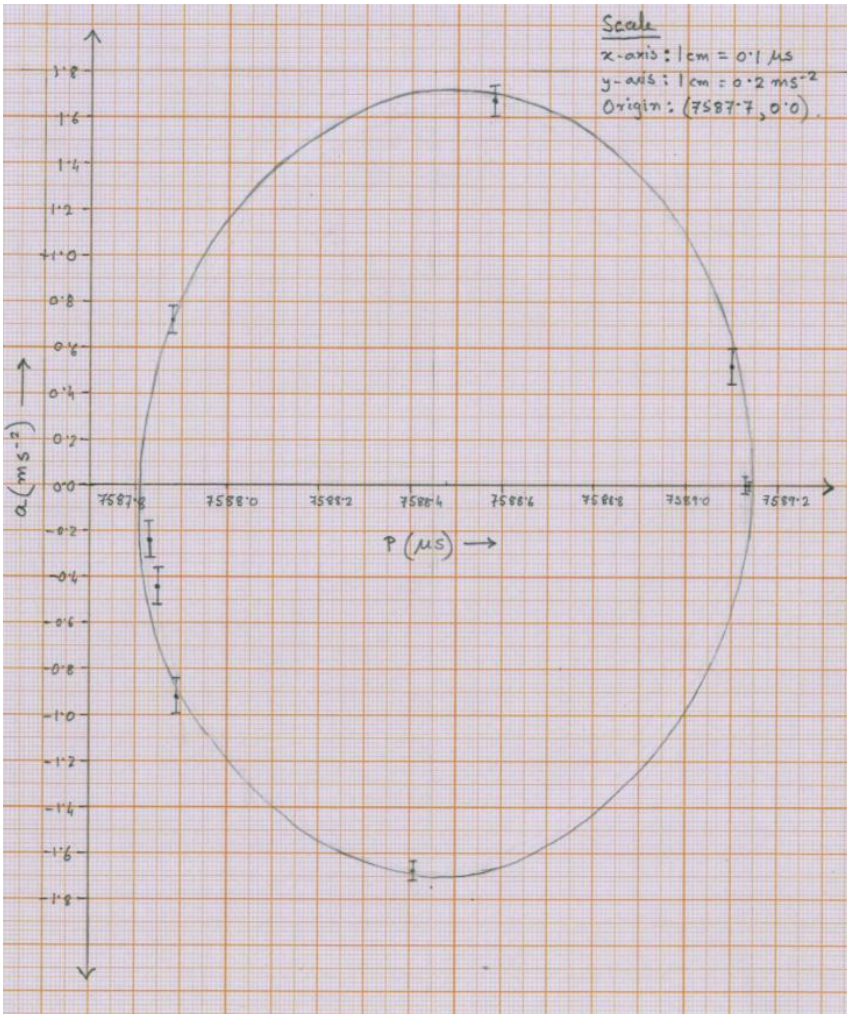
By plotting $a(\phi)$ as a function of $P(\phi)$, we can obtain a parametric curve. As evident from the relations above, this curve in the period-acceleration plane is an ellipse.

In this problem, we estimate the intrinsic spin period, P_0, the orbital period, P_B, and the orbital radius, r, by an analysis of this data set, assuming a circular orbit.

(D1.1) Plot the data, including error bars, in the period-acceleration plane (mark your graph as "D1.1"). **7**

Solution:

Graph Number : D1.1



- **Plot uses more than 50% of graph paper: 0.5**
- **Axes labels (P and a): 0.5**
- **Dimensions of axes: 0.5**
- **Ticks and values on axes (or scale written explicitly): 0.5**
- **Points correctly plotted:**

Points plotted	**9**	**8**	**7**	**6**	**5**	**< 5**
Marks given	**4.0**	**3.5**	**3.0**	**2.0**	**1.0**	**0**

Correctness of points: deduction of 0.5 for each wrong point.

- **Errorbars on points (at least 5): 1.0**

(D1.2) Draw an ellipse that appears to be a best fit to the data (on the same graph "D1.1"). **2**

Solution:
See above

- **Elliptical curve with visual best fit: 1.0**
- **Curve symmetric about $a = 0$ line: 0.5**

- **Curve symmetric about some value of P ($P \approx 7588.48$): 0.5**

(D1.3) From the plot, estimate P_0, P_t and a_t, including error margins. 7

Solution:
Values are determined from lengths of axes of ellipse and mid-point of P-axis.
Error margins may be determined by estimating extreme ellipses covering the points with errorbars. Any reasonable method of estimating error margins will be accepted.

$2P_t = (1.34 \pm 0.04)\,\mu s$ [(13.4 ± 0.4) cm on graph]

$\boxed{P_t = (0.67 \pm 0.02)\,\mu s}$ **2.0**

$\boxed{P_0 = (7588.48 \pm 0.02)\,\mu s}$ **2.0**

$2a_t = (3.42 \pm 0.12)\,m\,s^{-2}$ [(17.1 ± 0.6) cm on graph] **3.0**

$\boxed{a_t = (1.71 \pm 0.06)\,m\,s^{-2}}$

- **Marking table:**

Parameter	Half credit	Full credit		Half credit
	Minimum	Minimum	Maximum	Maximum
$P_t(\mu s)$	**0.59**	**0.63**	**0.71**	**0.75**
$\delta P_t(\mu s)$	**0.01**	**0.02**	**0.04**	**0.05**
$P_0(\mu s)$	**7588.38**	**7588.43**	**7588.53**	**7588.58**
$\delta P_0(\mu s)$	**0.01**	**0.02**	**0.04**	**0.05**
$a_t(m\,s^{-2})$	**1.61**	**1.65**	**1.77**	**1.81**
$\delta a_t(m\,s^{-2})$	**0.04**	**0.05**	**0.07**	**0.08**

- **Wrong values due to wrong/poor plot/fit in (D1.1) and (D1.2) WILL BE penalised.**
- **Error estimation is based on graph drawing. Quoted values correspond to the envelope of possible ellipses drawn to include all points with errorbars. Any reasonable method to estimate error to be given credit.**

(D1.4) Write expressions for P_B and r in terms of P_0, P_t, a_t. 4

Solution:
We can easily recover the orbital period (P_B) and the radius of the orbit (r) in a circular orbit:

$$a_t = \frac{4\pi^2}{P_B^2} r$$

$$\therefore r = \frac{P_B^2 a_t}{4\pi^2}$$ **1.0**

$$P_t = \frac{2\pi P_0}{P_B} \times \frac{r}{c}$$

$$= \frac{2\pi P_0}{P_B c} \times \frac{P_B^2 a_t}{4\pi^2} = \frac{P_0 P_B a_t}{2\pi c}$$

$$\therefore \boxed{P_\mathrm{B} = \frac{P_\mathrm{t}}{P_0}\frac{2\pi c}{a_\mathrm{t}}}$$ **1.0**

$$r = \frac{a_\mathrm{t}}{4\pi^2}\left(\frac{P_\mathrm{t}}{P_0}\frac{2\pi c}{a_\mathrm{t}}\right)^2$$ **1.0**

$$\therefore \boxed{r = \left(\frac{P_\mathrm{t}}{P_0}\right)^2\frac{c^2}{a_\mathrm{t}}}$$ **1.0**

Alternative algebraic routes accepted. Each of P_B and r carry 2.0 marks.

(D1.5) Calculate approximate value of P_B and r based on your estimations made in (D1.3), including error margins. **6**

Solution:

$$P_\mathrm{B} = \frac{P_\mathrm{t}}{P_0}\frac{2\pi c}{a_\mathrm{t}}$$

$$= \frac{0.67}{7588.48} \times \frac{2\pi \times 2.998 \times 10^8}{1.71}\mathrm{s}$$

$$= 96\,260\,\mathrm{s} = 1.125\,70\,\mathrm{d}$$ **1.0**

$$\Delta P_\mathrm{B} = P_\mathrm{B}\sqrt{\left(\frac{\Delta P_\mathrm{t}}{P_\mathrm{t}}\right)^2 + \left(\frac{\Delta P_0}{P_0}\right)^2 + \left(\frac{\Delta a_\mathrm{t}}{a_\mathrm{t}}\right)^2}$$ **1.0**

$$= 1.12570 \times \sqrt{\left(\frac{0.02}{0.67}\right)^2 + \left(\frac{0.02}{7588.48}\right)^2 + \left(\frac{0.06}{1.71}\right)^2}\,\mathrm{d}$$

$$= 1.12570 \times 0.0461 \simeq 0.052\,\mathrm{d}$$

$$\therefore \boxed{P_\mathrm{B} = (1.13 \pm 0.05)\,\mathrm{d}}$$ **1.0**

$$r = \left(\frac{P_\mathrm{t}}{P_0}\right)^2\frac{c^2}{a_\mathrm{t}}$$

$$= \left(\frac{0.67}{7588.48}\right)^2 \times \frac{\left(2.998 \times 10^8\right)^2}{1.71}\mathrm{m}$$

$$\therefore r = 4.097\,39 \times 10^8\,\mathrm{m} = 2.738\,90 \times 10^{-3}\,\mathrm{AU}$$ **1.0**

$$\Delta r = r\sqrt{\left(\frac{2\Delta P_\mathrm{t}}{P_\mathrm{t}}\right)^2 + \left(\frac{2\Delta P_0}{P_0}\right)^2 + \left(\frac{\Delta a_\mathrm{t}}{a_\mathrm{t}}\right)^2}$$ **1.0**

$$= 2.738\,90 \times 10^{-3} \times \sqrt{\left(\frac{2 \times 0.02}{0.67}\right)^2 + \left(\frac{2 \times 0.02}{7588.48}\right)^2 + \left(\frac{0.06}{1.71}\right)^2}\,\mathrm{AU}$$

$$= 2.738\,90 \times 10^{-3} \times 0.069\,25\,\mathrm{AU} \simeq 0.19 \times 10^{-3}\,\mathrm{AU}$$

$$\boxed{r = (2.74 \pm 0.19) \times 10^{-3}\,\mathrm{AU}}$$ **1.0**

Errors in P_B and r can be also estimated as maximum possible (worst case) error. In such case, errors would be about 1.5 times the standard error calculated above ($\delta P_\mathrm{B} = 0.07, \delta r = 0.25$).

(D1.6) Calculate orbital phase, ϕ, corresponding to the epochs of the following five observations **4** in the above table: data rows 1, 4, 6, 8, 9.

Solution:

Using these newly determined orbital parameters, we can calculate the angular orbital phase for each data point, i.e., for each pair of acceleration and period measured (P, a).

$$\phi = \tan^{-1}\left(-\frac{a}{a_t}\frac{P_t}{P - P_0}\right)$$

Care has to be taken to choose the value of the phase from among $\phi, \pi \pm \phi, 2\pi - \phi$, depending on the sign of $\cos\phi$ and $\sin\phi$.

Sr. no.	T (tMJD)	P (µs)	a ($\mathrm{m\,s^{-2}}$)	ϕ
1	5740.654	7587.8889	−0.92	148.62°
4	5746.675	7588.5810	+1.67	278.77°
6	5983.932	7587.8552	−0.44	164.57°
8	6040.857	7589.1350	+0.00	0.00°
9	6335.904	7589.1358	+0.00	0.00°

- **Credit for each correct value: 1.0 for first three, 0.5 for last two.**
- **Credit for $\pi \pm \phi$ or $2\pi - \phi$ is 0.5 per value.**
- **All values wrong due to wrong expression for ϕ gets a maximum of 1.0 mark.**
- **Values in radians accepted.**

(D1.7) Refine the estimate of the orbital period, P_B, using the results in part (D1.6) in the following way:

(D1.7a) First determine the initial epoch, T_0, which corresponds to the nearest epoch of zero phase before the first observation. **2**

Solution:

$$\frac{T_1 - T_0}{P_B} = \frac{\phi_1}{2\pi} \Rightarrow T_0 = T_1 - \frac{\phi_1}{2\pi}P_B$$ **1.0**

$$T_0 = 5740.654 - \frac{148.62°}{360°} \times 1.12570\,\mathrm{tMJD}$$

$$\boxed{T_0 = 5740.189\,\mathrm{tMJD}}$$ **1.0**

Tolerance: ± 0.002 tMJD. Using P_0 instead of P_B gets zero.

(D1.7b) The expected time, T_{calc}, of the estimated phase of each observation is given by **7**

$$T_{calc} = T_0 + \left(n + \frac{\phi}{360°}\right)P_B,$$

where n is the number of full cycle of orbital phases elapsed between T_0 and T_{calc}. Estimate n and T_{calc} for each of the five observations in part (D1.6). Note down difference T_{O-C} between observed T and T_{calc}. Enter these calculations in the table given in the Summary Answersheet.

Solution:

$$T_{\rm calc} = T_0 + \left(n + \frac{\phi}{360°}\right) P_{\rm B}$$

where n = Integer part of $[(T - T_0)/P_{\rm B}]$.

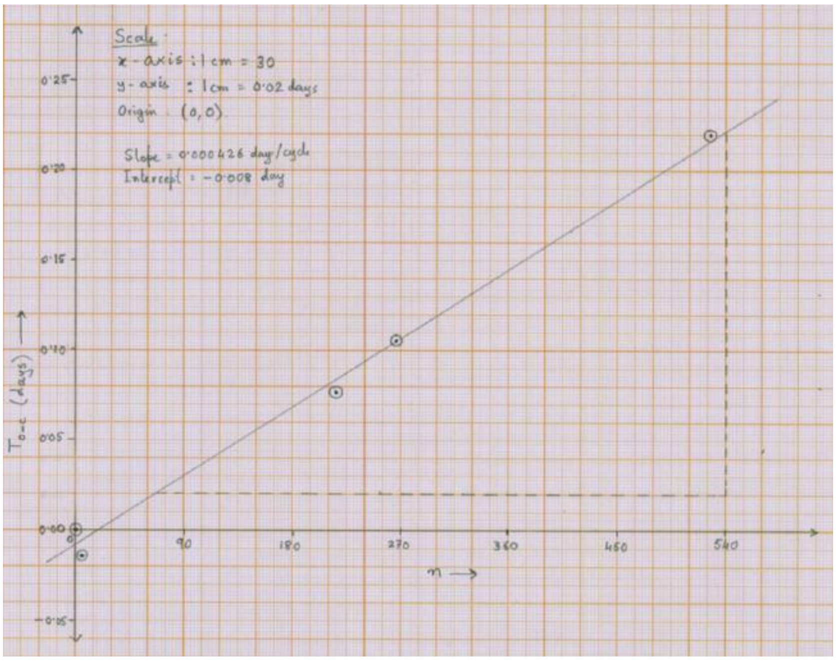
Sr. No.	T (tMJD)	ϕ	n	$T_{\rm calc}$ (MJD)	$T_{\rm O-C}$ (days)
1	5740.654	148.62°	0	5740.654	0.000
4	5746.675	278.77°	5	5746.689	−0.014
6	5983.932	164.57°	216	5983.855	0.077
8	6040.857	0.00°	267	6040.751	0.106
9	6335.904	0.00°	529	6335.684	0.220

Deduction for each wrong/missing value of n, $T_{\rm calc}$ and $T_{\rm O-C}$: 0.5
No double penalty in one row.

(D1.7c) Plot $T_{\rm O-C}$ against n (mark your graph as "D1.7"). **4**

Solution:

Graph Number : D1.7



- **Plot uses more than 50% of graph paper: 0.5**
- **Axes labels ($T_{\rm O-C}$ and n) including dimensions: 0.5**
- **Ticks and values on axes (or scale written explicitly): 0.5**
- **Points correctly plotted: 0.5 for each point**
- **Goodness of linear fit credited in next part**

(D1.7d) Determine the refined values of the initial epoch, $T_{0,\rm r}$, and the orbital period, $P_{\rm B,r}$. **7**

Solution:
A linear fit to the plot of $T_{\mathrm{O-C}}$ vs n gives the offset of period per cycle (slope) and the shift in the zero-phase point (intercept). **2.0**

This concept, which may be evident in the subsequent calculation, gains the credit, explicit statement is not necessary.

From a linear fit,

$$\text{Slope} = 0.000\,43\,\mathrm{d/n} \qquad \text{Intercept} = -0.010\,\mathrm{d}$$ **3.0**

- **Credit for good visual linear fit: 1.0**
- **Correct values of slope and intercept: 1.0 each**
- **Tolerance:** ± 0.00002 **in slope and** ± 0.002 **in intercept.**

$$T_{0,\mathrm{r}} = 5740.189 - 0.010 = \boxed{5740.179\,\mathrm{tMJD}}$$ **1.0**

$$P_{\mathrm{B,r}} = (1.12570 + 0.00043)\,\mathrm{d}$$
$$= 1.126\,13\,\mathrm{d}$$

$$\boxed{P_{\mathrm{B}} = 1.1261\,\mathrm{d}}$$ **1.0**

Incorrect sign of correction applied carries penalty of 0.5 for each quantity.

(D2) **Distance to the Moon**

Geocentric ephemerides of the Moon for September 2015 are given in the form of a table. Each reading was taken at 00:00 UT.

Date	R.A. (α)			Dec. (δ)			Angular Size (θ)	Phase (ϕ)	Elongation
	h	m	s	°	′	″	″		of Moon
Sep 01	0	36	46.02	3	6	16.8	1991.2	0.927	148.6° W
Sep 02	1	33	51.34	7	32	26.1	1974.0	0.852	134.7° W
Sep 03	2	30	45.03	11	25	31.1	1950.7	0.759	121.1° W
Sep 04	3	27	28.48	14	32	4.3	1923.9	0.655	107.9° W
Sep 05	4	23	52.28	16	43	18.2	1896.3	0.546	95.2° W
Sep 06	5	19	37.25	17	55	4.4	1869.8	0.438	82.8° W
Sep 07	6	14	19.23	18	7	26.6	1845.5	0.336	70.7° W
Sep 08	7	7	35.58	17	23	55.6	1824.3	0.243	59.0° W
Sep 09	7	59	11.04	15	50	33.0	1806.5	0.163	47.5° W
Sep 10	8	49	0.93	13	34	55.6	1792.0	0.097	36.2° W
Sep 11	9	37	11.42	10	45	27.7	1780.6	0.047	25.1° W
Sep 12	10	23	57.77	7	30	47.7	1772.2	0.015	14.1° W
Sep 13	11	9	41.86	3	59	28.8	1766.5	0.001	3.3° W
Sep 14	11	54	49.80	0	19	50.2	1763.7	0.005	7.8° E
Sep 15	12	39	50.01	-3	20	3.7	1763.8	0.026	18.6° E
Sep 16	13	25	11.64	-6	52	18.8	1767.0	0.065	29.5° E
Sep 17	14	11	23.13	-10	9	4.4	1773.8	0.120	40.4° E
Sep 18	14	58	50.47	-13	2	24.7	1784.6	0.189	51.4° E
Sep 19	15	47	54.94	-15	24	14.6	1799.6	0.270	62.5° E
Sep 20	16	38	50.31	-17	6	22.8	1819.1	0.363	73.9° E
Sep 21	17	31	40.04	-18	0	52.3	1843.0	0.463	85.6° E
Sep 22	18	26	15.63	-18	0	41.7	1870.6	0.567	97.6° E
Sep 23	19	22	17.51	-17	0	50.6	1900.9	0.672	110.0° E
Sep 24	20	19	19.45	-14	59	38.0	1931.9	0.772	122.8° E
Sep 25	21	16	55.43	-11	59	59.6	1961.1	0.861	136.2° E
Sep 26	22	14	46.33	-8	10	18.3	1985.5	0.933	150.0° E
Sep 27	23	12	43.63	-3	44	28.7	2002.0	0.981	164.0° E
Sep 28	0	10	48.32	0	58	58.2	2008.3	1.000	178.3° E
Sep 29	1	9	5.89	5	38	54.3	2003.6	0.988	167.4° W
Sep 30	2	7	39.02	9	54	16.1	1988.4	0.947	153.2° W

The composite graphic[1] below shows multiple snapshots of the Moon taken at different times during the total lunar eclipse, which occurred in this month. For each shot, the centre of frame was coinciding with the central north-south line of umbra.

For this problem, assume that the observer is at the centre of the Earth and angular size refers to angular diameter of the relevant object / shadow.

(D2.1) In September 2015, apogee of the lunar orbit is closest to New Moon / First Quarter / Full Moon / Third Quarter. Tick the correct answer in the Summary Answersheet. No justification for your answer is necessary. **3**

> **Solution:**
> From the table we see that the angular size of Moon is smallest close to the New Moon day. Thus, the answer is New Moon. **3.0**
> **Justification is NOT necessary for full credit.**

(D2.2) In September 2015, the ascending node of lunar orbit with respect to the ecliptic is closest to New Moon / First Quarter / Full Moon / Third Quarter. Tick the correct answer in the Summary Answersheet. No justification for your answer is necessary. **4**

> **Solution:**
> As there is an eclipse happening in this month, the lunar nodes are close to Full Moon day and New Moon day. Next we notice that lowest declination of Moon is just 18°. This means that after the New Moon day, the orbit of Moon is above the ecliptic. In other words, the ascending node is near the New Moon. **4.0**
> **Justification is NOT necessary for full credit.**

(D2.3) Estiamte the eccentricity, e, of the lunar orbit from the given data. **4**

> **Solution:**
> The largest angular size of the Moon in the ephemerides is 2008.3″ and the smallest angular size is 1763.7″. The distance is inversely proportional to the angular size. Hence ratio of distance at perigee to the distance at apogee is:
>
> $$\text{Ratio} = \frac{r_{\text{perigee}}}{r_{\text{apogee}}} = \frac{\cancel{a}}{1+e} \times \frac{1-e}{\cancel{a}}$$ **2.0**

[1]Credit: NASA's Scientific Visualization Studio

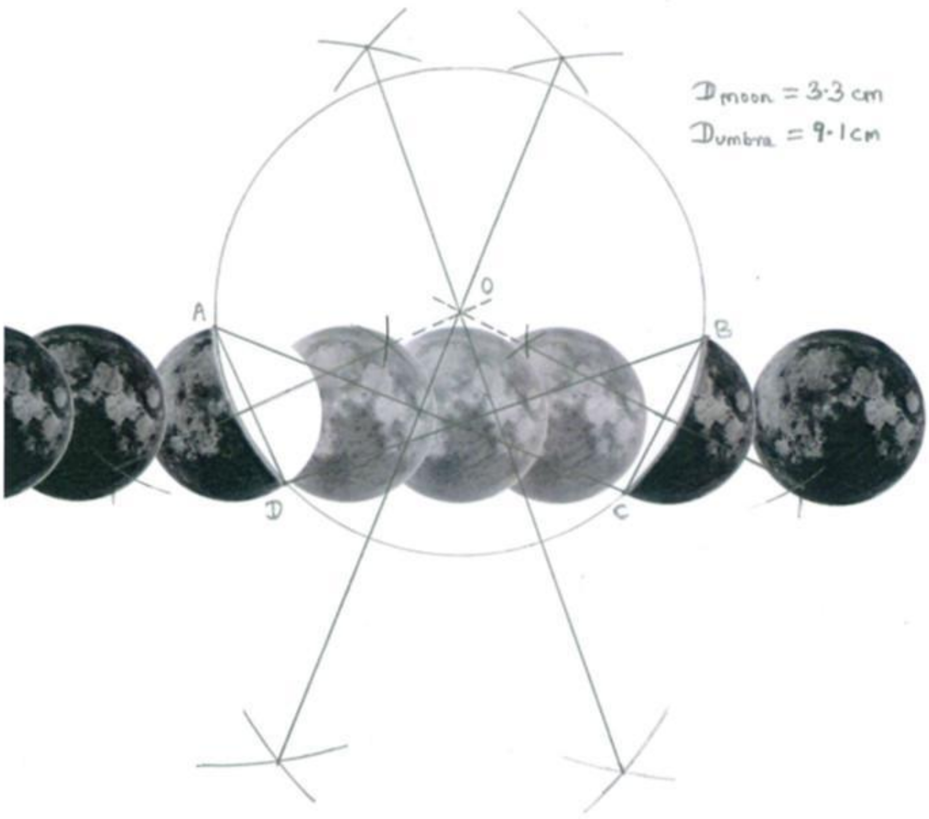
$$\therefore \frac{1-e}{1+e} = \frac{1763.7}{2008.3} = 0.87821$$
$$\therefore e = \frac{1-0.87821}{1+0.87821} = 0.064846$$
$$\boxed{e \simeq 0.065}$$ **2.0**

Rounding off is done to account for the fact that our data are not continuous, hence exact angular sizes at perigee and apogee are not known. A non-rounded answer will also receive full credit.

(D2.4) Estimate the angular size of the umbra, θ_{umbra}, in terms of the angular size of the Moon, θ_{Moon}. Show your working on the image given on the backside of the Summary Answersheet. **8**

Solution:

The following construction is shown. **5.0**



Only two chords are necessary to determine the centre.

The credit is divided in two parts for the drawing:

- **Realisation that centre of umbra circle needs to be determined to find θ_{umbra}: 1.5**
- **Accurate determination of centre of umbra circle by geometric construction: 2.5**
  Determination of centre of hand-drawn circle: maximum 1.5

- **Measuring diameters of umbra and Moon: 0.5 each**

By estimating approximate centre of the shadow in the image, we find out,

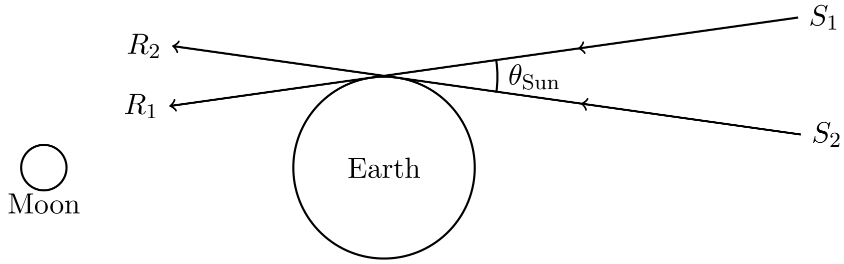
$$\frac{\theta_{\text{umbra}}}{\theta_{\text{Moon}}} = \frac{d_{\text{umbra}}}{d_{\text{Moon}}}$$ **2.0**

$$= \frac{9.1}{3.3} = 2.76$$

$$\therefore \boxed{\theta_{\text{umbra}} \simeq 2.76\theta_{\text{Moon}}}$$ **1.0**

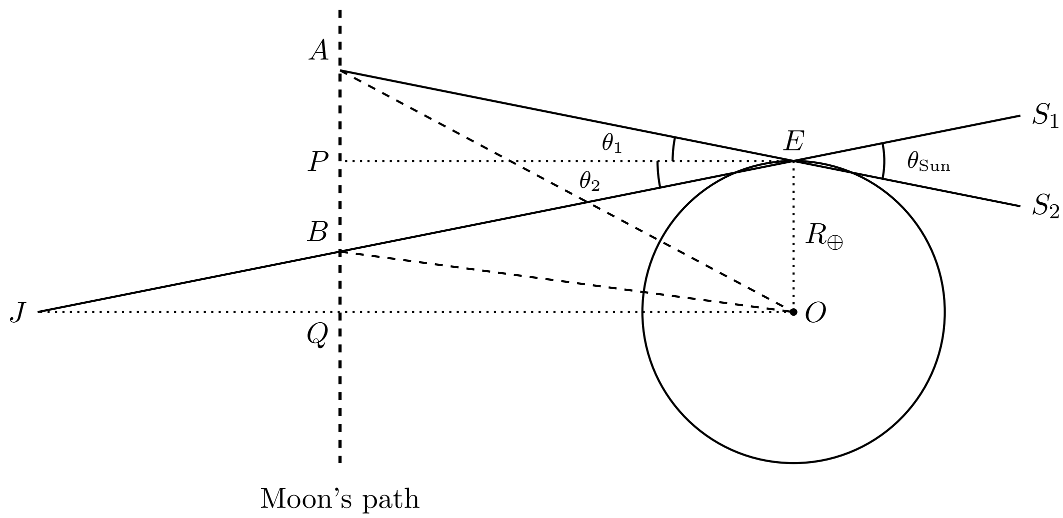
Acceptable range: $\pm$**0.10.**

(D2.5) The angle subtended by the Sun at Earth on the day of the lunar eclipse is known to be $\theta_{\text{Sun}} = 1915.0''$. In the figure below, S_1R_1 and S_2R_2 are rays coming from diametrically opposite ends of the solar disk. The figure is not to scale. **9**



Calculate the angular size of the penumbra, θ_{penumbra}, in terms of θ_{Moon}. Assume the observer to be at the centre of the Earth.

Solution:
The following diagram needs to be drawn.



Angular size of umbra is $\theta_{\text{umbra}} = 2\measuredangle BOQ$
Angular size of penumbra is $\theta_{\text{penumbra}} = 2\measuredangle AOQ$
We have

$$QA = QP + PA$$
$$= OE + PE \tan\theta_1$$

$$\approx R_\oplus + d_{\text{Moon}}\theta_1 \qquad \text{since } PA \ll PE$$

$$\approx R_\oplus + d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2} \qquad \text{since } \theta_1 \approx \theta_2 \approx \theta_{\text{Sun}}/2$$ **2.0**

and

$$QB = QP - PB$$
$$= OE - PE\tan\theta_2$$
$$\approx R_\oplus - d_{\text{Moon}}\theta_2$$
$$\approx R_\oplus - d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2}$$ **2.0**

$$\therefore \theta_{\text{penumbra}} = 2\measuredangle AOQ = 2\tan^{-1}\left(\frac{QA}{OQ}\right) \approx 2\frac{QA}{OQ} \qquad \text{since } QA \ll OQ$$

$$= 2\frac{R_\oplus + d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2}}{d_{\text{Moon}}} = \frac{2R_\oplus}{d_{\text{Moon}}} + \theta_{\text{Sun}}$$ **1.0**

and

$$\theta_{\text{umbra}} = 2\measuredangle BOQ = 2\tan^{-1}\left(\frac{QB}{OQ}\right) \approx 2\frac{QB}{OQ}$$

$$= 2\frac{R_\oplus - d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2}}{d_{\text{Moon}}} = \frac{2R_\oplus}{d_{\text{Moon}}} - \theta_{\text{Sun}}$$ **1.0**

Subtracting,

$$\theta_{\text{penumbra}} - \theta_{\text{umbra}} = 2\theta_{\text{Sun}} \Rightarrow \theta_{\text{penumbra}} = \theta_{\text{umbra}} + 2\theta_{\text{Sun}}$$ **1.0**

We have,

$$\theta_{\text{umbra}} = 2.76\theta_{\text{Moon}} \qquad \text{and } \theta_{\text{Sun}} = 1915.0''$$

From the given data, $\theta_{\text{Moon}} = 2008.3''$. **1.0**

Therefore,

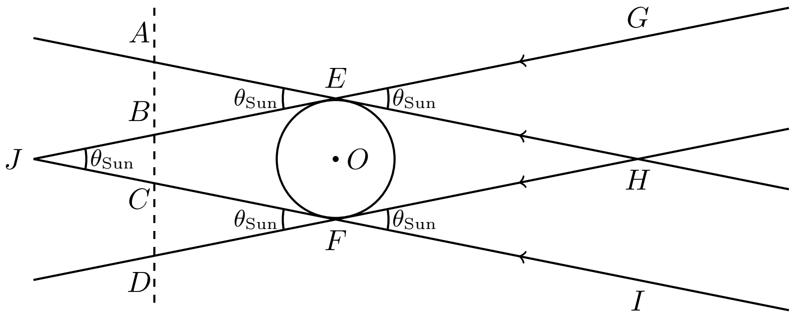
$$\theta_{\text{penumbra}} = 2.76\theta_{\text{Moon}} + 2\frac{1915.0}{2008.3}\theta_{\text{Moon}}$$

$$\boxed{\theta_{\text{penumbra}} = 4.67\theta_{\text{Moon}}}$$ **1.0**

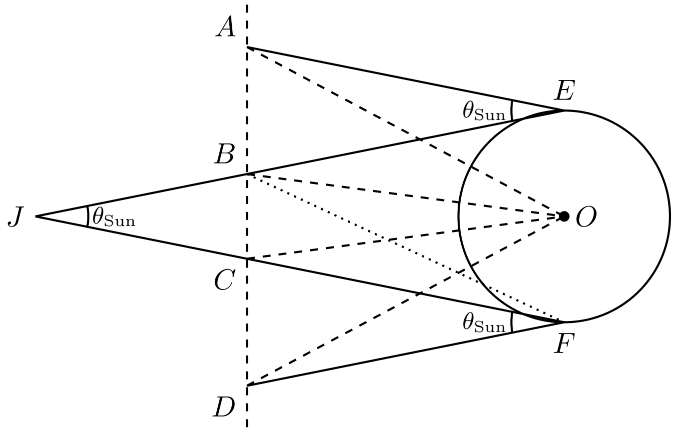
Acceptable range: $4.57\theta_{\text{Moon}}$ to $4.77\theta_{\text{Moon}}$.

Alternative solution:

In the figure below, rays HEA and IFC are coming from one edge of solar disk and rays HFD and GEB are coming from the opposite edge. The observer (O) is assumed to be at the centre of the Earth. The Moon travels along the path $ABCD$ during the course of eclipse.



Moon's path



Moon's path

From figure,

$$\measuredangle AEB = \measuredangle GEH = \measuredangle HFI = \measuredangle DFC = \measuredangle EJF = \theta_{\text{Sun}} \quad \textbf{3.0}$$

$$\theta_{\text{umbra}} = \measuredangle BOC = 2.76\theta_{\text{Moon}} \quad \textbf{1.0}$$

$$\theta_{\text{penumbra}} = \measuredangle AOD \quad \textbf{1.0}$$

$$\measuredangle AOD = \measuredangle AOB + \measuredangle BOC + \measuredangle COD$$

$$\measuredangle AOB = \measuredangle AEB \quad \textbf{1.0}$$

$$\measuredangle COD = \measuredangle CFD \quad \textbf{1.0}$$

$$\theta_{\text{penumbra}} \simeq \measuredangle AEB + \theta_{\text{umbra}} + \measuredangle CFD$$

$$= 2\theta_{\text{Sun}} + 2.76\theta_{\text{Moon}} = 2 \times 1915.0'' + 2.76 \times 2008.3''$$

$$\boxed{\theta_{\text{penumbra}} = 9372.9'' = 4.67\theta_{\text{Moon}}} \quad \textbf{2.0}$$

(D2.6) Let θ_{Earth} be angular size of the Earth as seen from the centre of the Moon. Calculate the angular size of the Moon, θ_{Moon}, as would be seen from the centre of the Earth on the eclipse day in terms of θ_{Earth}. **5**

Solution:

From the Moon,

$$\theta_{\text{Earth}} = \frac{2R_{\oplus}}{d_{\text{Moon}}} \quad \textbf{1.0}$$

From part (D2.5),

$$\theta_{\text{umbra}} + \theta_{\text{penumbra}} = 2\theta_{\text{Earth}}$$ **2.0**

$$\therefore \theta_{\text{Earth}} = \frac{\theta_{\text{umbra}} + \theta_{\text{penumbra}}}{2} = \frac{2.76 + 4.67}{2}\theta_{\text{Moon}} = 3.72\theta_{\text{Moon}}$$

$$\boxed{\theta_{\text{Moon}} = = 0.269\theta_{\text{Earth}}}$$ **2.0**

<u>Alternative solution</u>:

Let us say that the Moon is at position of B. Thus, angular size of Earth as seen from this position will be, (see figure in the previous part)

$$\theta_{\text{Earth}} = \measuredangle EBF = \measuredangle BFD$$ **2.0**

$$= \measuredangle BFC + \measuredangle CFD$$

$$\simeq \theta_{\text{umbra}} + \theta_{\text{Sun}}$$ **1.0**

The angular size of the Full Moon on 28 September as seen in the table is $2008.3''$.

$$\theta_{\text{Earth}} = 2.76 \times 2008.3'' + 1915.0'' = 7453.0''$$

$$\boxed{\theta_{\text{Moon}} = 0.269\theta_{\text{Earth}} = \frac{\theta_{\text{Earth}}}{3.72}}$$ **2.0**

(D2.7) Estimate the radius of the Moon, R_{Moon}, in km from the results above. **3**

Solution:

Thus, the radius of Moon will be,

$$R_{\text{Moon}} = \frac{R_{\oplus}}{3.72}$$

$$R_{\text{Moon}} = \frac{6371}{3.72}$$ **1.0**

$$\boxed{R_{\text{Moon}} \simeq 1713\,\text{km}}$$ **2.0**

Acceptable range: $\pm 20\,\text{km}$**.**

(D2.8) Estimate the shortest distance, r_{perigee}, and the farthest distance, r_{apogee}, to the Moon. **4**

Solution:

The shortest and longest distances will be,

$$r_{\text{perigee}} = \frac{2 \times 1713 \times 206265}{2008.3}$$

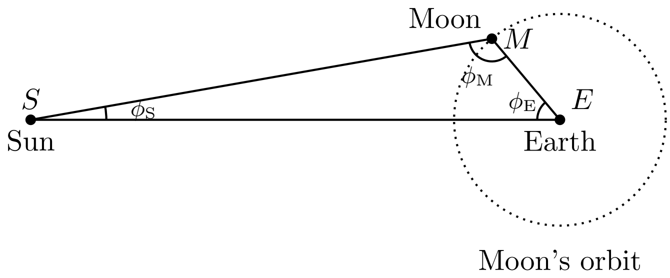
$$\boxed{r_{\text{perigee}} = 3.52 \times 10^5\,\text{km}}$$ **2.0**

$$r_{\text{apogee}} = \frac{2 \times 1713 \times 206265}{1763.7}$$

$$\boxed{r_{\text{apogee}} = 4.01 \times 10^5\,\text{km}}$$ **2.0**

(D2.9) Use appropriate data from September 10 to estimate the distance, d_{Sun}, to the Sun from the Earth. **10**

Solution:



On September 10, phase of Moon is 0.097 and elongation of Moon is 36.2°. Angular size of the Moon on this day is 1792.0″. Therefore, distance to Moon (from Earth) on September 10 is

$$d_{\text{Moon},10} = \frac{2 \times 1713 \times 206265}{1792.0}$$
$$= 3.94 \times 10^5 \text{ km} \quad \textbf{2.0}$$

$$\text{Let } \measuredangle EMS = \phi_M$$
$$\measuredangle ESM = \phi_S$$
$$\measuredangle SEM = \phi_E$$
$$\therefore \phi_E = 36.2^\circ \quad \textbf{1.0}$$
$$\text{phase} = \frac{1 + \cos\phi_M}{2} \quad \textbf{2.0}$$
$$\therefore \phi_M = \cos^{-1}(2 \times \text{ phase } - 1)$$
$$= \cos^{-1}(2 \times 0.097 - 1) = \cos^{-1}(-0.806)$$
$$= 143.71^\circ \quad \textbf{2.0}$$
$$\phi_S = 180^\circ - \phi_E - \phi_M$$
$$= 180^\circ - 36.2^\circ - 143.71^\circ$$
$$= 0.09^\circ \quad \textbf{1.0}$$

Now using sine rule,

$$\frac{d_{\text{Sun}}}{d_{\text{Moon},10}} = \frac{\sin\phi_M}{\sin\phi_S} \quad \textbf{2.0}$$
$$\therefore d_{\text{Sun}} = \frac{3.94 \times 10^8 \times \sin 143.71^\circ}{\sin 0.09^\circ}$$
$$\boxed{d_{\text{Sun}} = 1.48 \times 10^{11}\text{ m}} \quad \textbf{1.0}$$

(D3) **Type IA Supernovae**

Supernovae of type Ia are considered very important for the measurements of large extragalactic distances. The brightening and subsequent dimming of these explosions follow a characteristic light curve, which helps in identifying these as supernovae of type Ia.

Light curves of all type Ia supernovae can be fit to the same model light curve, when they are scaled appropriately. In order to achieve this, we first have to express the light curves in the reference frame of the host galaxy by taking care of the cosmological stretching/dilation of all observed time intervals, Δt_{obs}, by a factor of $(1+z)$. The time interval in the rest frame of the host galaxy is denoted by Δt_{gal}.

The rest frame light curve of a supernova changes by two magnitudes compared to the peak in a time interval Δt_0 after the peak. If we further scale the time intervals by a factor of s (i.e. $\Delta t_s = s\Delta t_{\text{gal}}$) such that the scaled value of Δt_0 is the same for all supernovae, the light curves turn out to have the same shape. It also turns out that s is related linearly to the absolute magnitude, M_{peak}, at the peak luminosity for the supernova. That is, we can write

$$s = a + bM_{\text{peak}} \ ,$$

where a and b are constants. Knowing the scaling factor, one can determine absolute magnitudes of supernovae at unknown distances from the above linear equation.

The table below contains data for three supernovae, including their distance moduli, μ (for the first two), their recession speed, cz, and their apparent magnitudes, m_{obs}, at different times. The time $\Delta t_{\text{obs}} \equiv t - t_{\text{peak}}$ shows number of days from the date at which the respective supernova reached peak brightness. The observed magnitudes have already been corrected for interstellar as well as atmospheric extinction.

Name	SN2006TD	SN2006IS	SN2005LZ
μ (mag)	34.27	35.64	
cz (km s^{-1})	4515	9426	12060
Δt_{obs} (days)	m_{obs} (mag)	m_{obs} (mag)	m_{obs} (mag)
−15.00	19.41	18.35	20.18
−10.00	17.48	17.26	18.79
−5.00	16.12	16.42	17.85
0.00	15.74	16.17	17.58
5.00	16.06	16.41	17.72
10.00	16.72	16.82	18.24
15.00	17.53	17.37	18.98
20.00	18.08	17.91	19.62
25.00	18.43	18.39	20.16
30.00	18.64	18.73	20.48

(D3.1) Compute Δt_{gal} values for all three supernovae, and fill them in the given blank boxes in the data tables on the BACK side of the Summary Answersheet. On a graph paper, plot the points and draw the three light curves in the rest frame (mark your graph as "D3.1"). **15**

Solution:

Redshifts for the three supernovae are $z_1 = 0.0151$, $z_2 = 0.0314$ and $z_3 = 0.0402$. **1.5**

Filling in the three tables (Δt_{gal}, third column) **3.5**

SN2006TD			
$\Delta t_{\rm obs}$	$m_{\rm obs}$	$\Delta t_{\rm gal}$	$\Delta t_{\rm s}$
(d)	(mag)	(d)	(d)
−15.00	19.41	−14.78	−20.00
−10.00	17.48	−9.85	−13.34
−5.00	16.12	−4.93	−6.67
0.00	15.74	0.00	0.00
5.00	16.06	4.93	6.67
10.00	16.72	9.85	13.34
15.00	17.53	14.78	20.00
20.00	18.08	19.70	26.67
25.00	18.43	24.63	33.34
30.00	18.64	29.56	40.01

SN2006IS			
$\Delta t_{\rm obs}$	$m_{\rm obs}$	$\Delta t_{\rm gal}$	$\Delta t_{\rm s}$
(d)	(mag)	(d)	(d)
−15.00	18.35	−14.54	−14.54
−10.00	17.26	−9.70	−9.70
−5.00	16.42	−4.85	−4.85
0.00	16.17	0.00	0.00
5.00	16.41	4.85	4.85
10.00	16.82	9.70	9.70
15.00	17.37	14.54	14.54
20.00	17.91	19.39	19.39
25.00	18.39	24.24	24.24
30.00	18.73	29.09	29.09

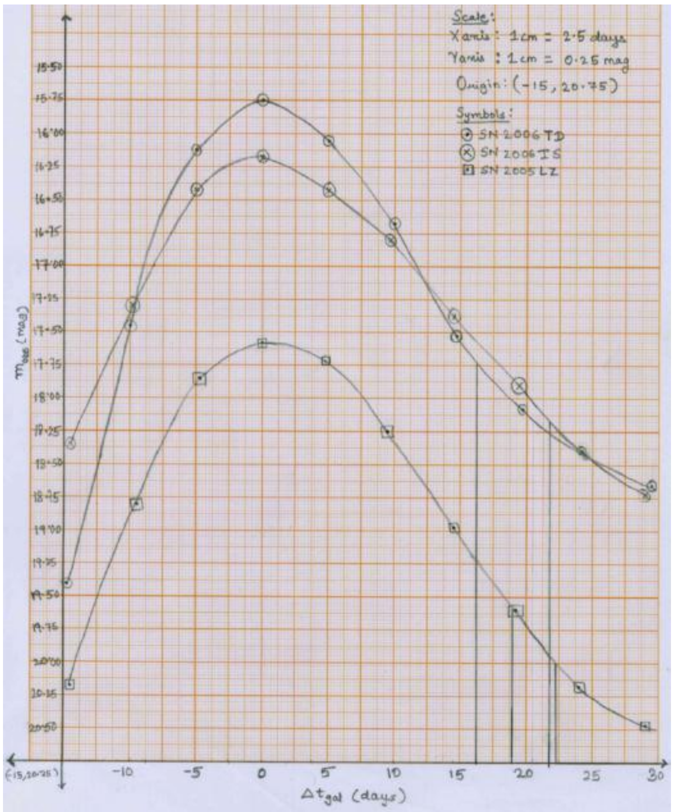
SN2005LZ			
$\Delta t_{\rm obs}$	$m_{\rm obs}$	$\Delta t_{\rm gal}$	$\Delta t_{\rm s}$
(d)	(mag)	(d)	(d)
−15.00	20.18	−14.42	−17.03
−10.00	18.79	−9.61	−11.35
−5.00	17.85	−4.81	−5.68
0.00	17.58	0.00	0.00
5.00	17.72	4.81	5.68
10.00	18.24	9.61	11.35
15.00	18.98	14.42	17.03
20.00	19.62	19.23	22.70
25.00	20.16	24.03	28.38
30.00	20.48	28.84	34.06

Full marks of 3.5 for all correct values.
Penalty for incorrect values (3×7 independent values):

Incorrect	**1-3**	**4-6**	**7-9**	**10-12**	**13-15**	**16-18**	**19-21**
Deduction	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**

The light curves in galaxy frame would appear as follows **10.0**

Graph Number: D3.1



- **Plot uses more than 50% of graph paper: 0.5**

- **Both axes labels (Δt_{gal} and m_{obs}) present: 0.5**
- **Both dimensions of axes (days and mag) present: 0.5**
- **Ticks and values on axes (or scale written explicitly): 0.5**
- **Points correctly plotted:**
  All points correctly plotted: 5.0
  Penalty for incorrect or missing points:

Incorrect	1	2-4	5-7	8-10	11-13	14-16	17-19	20-22	23-25	26-30
Deduction	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0

- **Smooth curve through points: 1.0 per curve**

(D3.2) Take the scaling factor, s_2, for the supernova SN2006IS to be 1.00. Calculate the scaling factors, s_1 and s_3, for the other two supernovae SN2006TD and SN 2005LZ, respectively, by calculating Δt_0 for them. **5**

Solution:

From the graph D3.1, SN2006IS took 22.0 d to fade by 2 magnitudes.

That is, Δt_0(SN2006IS) = 22.0 d.

Similarly, Δt_0(SN2006TD) = 16.4 d.

And Δt_0(SN2005LZ) = 18.8 d.

Acceptable range: ± 1.0 **days** **3.0**

Thus, stretching factors for these two supernovae are

$$s_1 = \frac{22.2}{16.4} = 1.354$$

$$s_3 = \frac{22.2}{18.8} = 1.181$$ **2.0**

(D3.3) Compute the scaled time differences, Δt_{s}, for all three supernovae. Write the values for Δt_{s} in the same data tables on the Summary Answersheet. On another graph paper, plot the points and draw 3 light curves to verify that they now have an identical profile (mark your graph as "D3.3"). **14**

Solution:

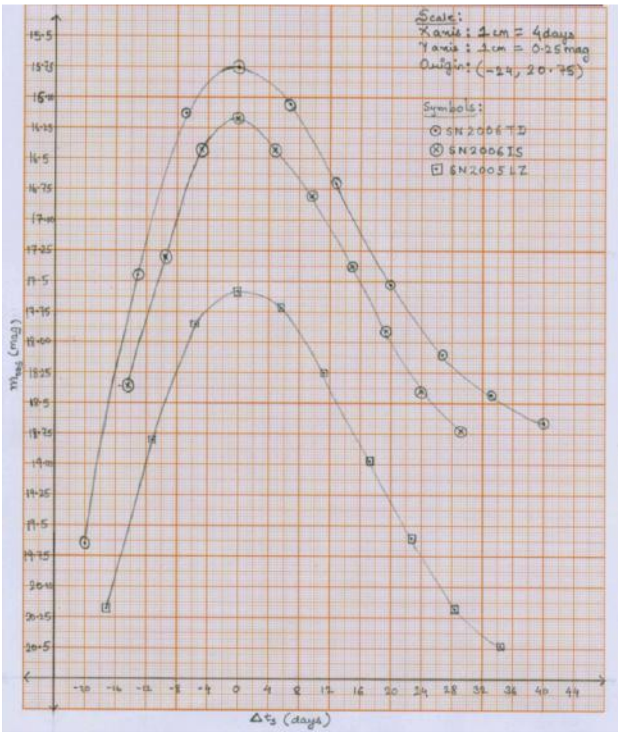
Filling the scaled values in the fourth column of the table (Δt_{s} in table above) **3.5**

Full marks of 3.5 for all correct values.
Penalty for incorrect values (3×7 independent values):

Incorrect	1-3	4-6	7-9	10-12	13-15	16-18	19-21
Deduction	0.5	1.0	1.5	2.0	2.5	3.0	3.5

The scaled light curves would appear as follows,

Graph Number: D3.3



10.5

- **Plot uses more than 50% of graph paper: 0.5**
- **Both axes labels (Δt_s and m_{obs}) present: 0.5**
- **Both dimensions of axes (days and mag) present: 0.5**
- **Ticks and values on axes (or scale written explicitly): 0.5**
- **Points correctly plotted:**
  All points correctly plotted: 5.0
  Penalty for incorrect or missing points:

Incorrect	1	2-4	5-7	8-10	11-13	14-16	17-19	20-22	23-25	26-30
Deduction	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0

- **Smooth curve through points: 1.0 per curve**
- **The curves should show identical profiles.** 0.5

(D3.4) Calculate the absolute magnitudes at peak brightness, $M_{\text{peak},1}$, for SN2006TD and $M_{\text{peak},2}$, for SN2006IS. Use these values to calculate a and b. 6

Solution:

To get a and b,

$$M_{\text{peak},1} = m_{\text{peak},1} - \mu_1 = 15.74 - 34.27\,\text{mag} = -18.53\,\text{mag}$$

$$M_{\text{peak},2} = m_{\text{peak},2} - \mu_2 = 16.17 - 35.64\,\text{mag} = -19.47\,\text{mag}$$

2.0

$$\therefore b = \frac{s_1 - s_2}{M_{\text{peak},1} - M_{\text{peak},2}} = \frac{1.354 - 1}{-18.53 - (-19.47)}\,\text{mag}^{-1} = \frac{0.354}{0.94}\,\text{mag}^{-1}$$

$\boxed{b = 0.3762\,\text{mag}^{-1}}$ **2.0**

$a = s_2 - bM_{\text{peak},2} = 1 - 0.3762 \times (-19.47) = 1 + 7.325$

$\boxed{a = 8.325}$ **2.0**

No penalty for missing mag^{-1} **in** b**.**

(D3.5) Calculate the absolute magnitude at peak brightness, $M_{\text{peak},3}$, and distance modulus, μ_3, for SN2005LZ. **4**

Solution:

$$s_3 = a + bM_{\text{peak},3}$$

$$\therefore M_{\text{peak},3} = \frac{s_3 - a}{b} = \frac{1.181 - 8.325}{0.3762}\,\text{mag} = \frac{-7.144}{0.3762}\,\text{mag}$$

$\boxed{M_{\text{peak},3} = -18.99\,\text{mag}}$ **2.0**

Distance modulus to SN2005LZ is

$$\mu_3 = m_{\text{peak},3} - M_{\text{peak},3} = 17.58 - (-18.99)\,\text{mag}$$

$\boxed{\mu_3 = 36.57\,\text{mag}}$ **2.0**

(D3.6) Use the distance modulus μ_3 to estimate the value of Hubble's constant, H_0. Further, estimate the characteristic age of the universe, T_{H}. **6**

Solution:

Distance to SN2005LZ is

$$d_3 = 10^{\left(\frac{\mu_3}{5}+1\right)}\text{pc} = 10^{\left(\frac{\mu_3}{5}+1-6\right)}\text{Mpc}$$

$$= 10^{\left(\frac{36.57}{5}-5\right)}\text{Mpc} = 10^{2.314}\text{Mpc}$$

$$\simeq 206\,\text{Mpc}$$

$$H_0 = \frac{cz_3}{d_3} = \frac{12060}{206}\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$$

$\boxed{H_0 = 58.5\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}}$ **4.0**

$$T_{\text{H}} = \frac{1}{H_0} = \frac{3.086 \times 10^{22}}{58.5 \times 10^3 \times 3.156 \times 10^7}\,\text{yr}$$

$\boxed{T_{\text{H}} = 16.7\,\text{Gyr}}$ **2.0**

Extra factor of $2/3$ **allowed in the value of** T_{H}**.**

(G1) A spacecraft of mass m and velocity $\vec{v}$ approaches a massive planet of mass M and orbital velocity $\vec{u}$, as measured by an inertial observer. We consider a special case, where the incoming trajectory of the spacecraft is designed in a way such that velocity vector of the planet does not change direction due to the gravitational boost given to the spacecraft. In this case, the amount of gravitational boost to the velocity the spacecraft can be roughly estimated using conservation laws by measuring asymptotic velocity of the spacecraft before and after the interaction and angle of approach of the spacecraft.

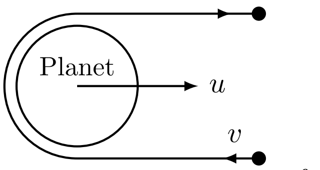


Figure 1

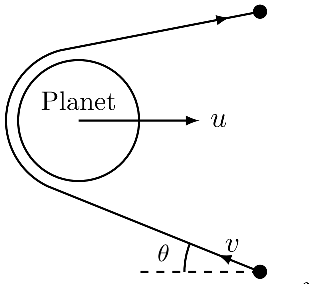


Figure 2

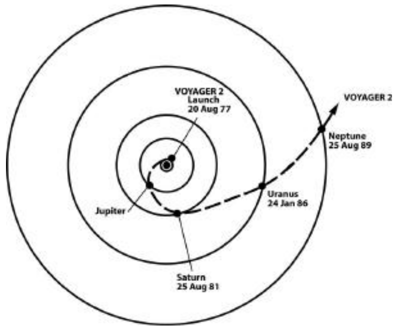


Figure 3

(G1.1) What will be the final velocity ($\vec{v_f}$) of the spacecraft, if $\vec{v}$ and $\vec{u}$ are exactly anti-parallel (see Figure 1). **3**

Solution:

Let $\vec{v_f}$ and $\vec{u_f}$ be the final velocity of the spacecraft and the planet respectively. As the planet For anti-parallel case, using conservation of linear momentum,

$$\begin{aligned} M\vec{u} + m\vec{v} &= M\vec{u_f} + m\vec{v_f} \\ \therefore Mu - mv &= Mu_f + mv_f \\ u_f &= u - \frac{m}{M}(v_f + v) \end{aligned}$$

Now, using conservation of energy,

$$\begin{aligned} Mu^2 + mv^2 &= Mu_f^2 + mv_f^2 \\ u^2 + \frac{m}{M}v^2 &= \left(u - \frac{m}{M}(v_f+v)\right)^2 + \frac{m}{M}v_f^2 \\ \cancel{u^2} + \cancel{\frac{m}{M}}v^2 &= \cancel{u^2} + \frac{m}{M}\cdot\cancel{\frac{m}{M}}(v_f+v)^2 - 2u\cancel{\frac{m}{M}}(v_f+v) + \cancel{\frac{m}{M}}v_f^2 \\ 0 &= \frac{m}{M}(v_f+v)^2 - 2u(v_f+v) + (v_f^2 - v^2) \\ 0 &= \frac{m}{M}(v_f+v)\cancel{(v_f+v)} - 2u\cancel{(v_f+v)} + \cancel{(v_f+v)}(v_f - v) \\ 0 &= \frac{m}{M}(v_f+v) - 2u + v_f - v \\ \therefore v_f\left(1 + \frac{m}{M}\right) &= 2u + \left(1 - \frac{m}{M}\right)v \\ v_f &= \frac{2u + \left(1 - \frac{m}{M}\right)v}{\left(1 + \frac{m}{M}\right)} \end{aligned}$$

Alternative solution in COM frame

(G1.2) Simplify the expression for the case where $m \ll M$. **1**

Solution:
If $m \ll M$,

$$v_f \approx 2u + v$$

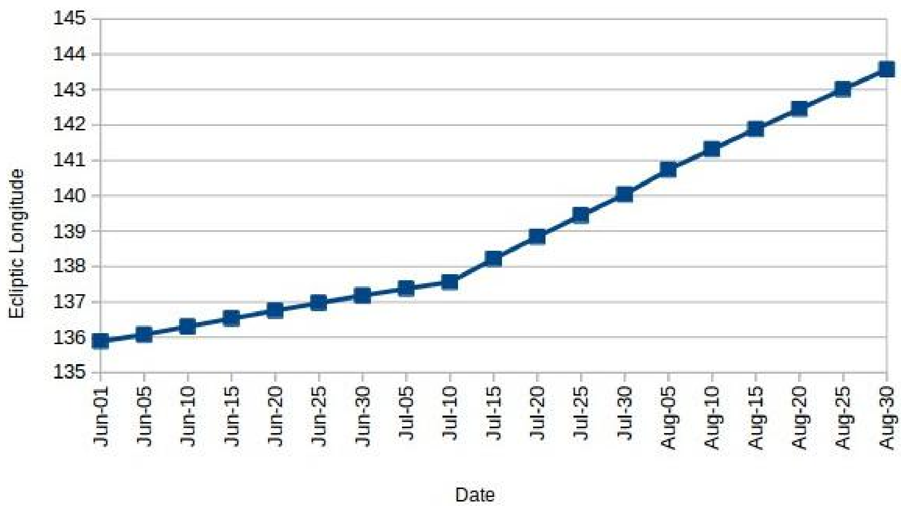
(G1.3) If angle between $\vec{v}$ and $-\vec{u}$ is θ and $m \ll M$ (see Figure 2), use results above to write expression for the magnitude of final velocity (v_f). **3**

Solution:
As velocity vector of the planet is not changing direction, there is no momentum transfer in direction perpendicular to $\vec{u}$. We will resolve $\vec{v}$ and $\vec{v_f}$ into components parallel and perpendicular to $\vec{u}$.

$$v_x = -v\cos\theta \qquad v_y = v\sin\theta$$
$$v_{f_x} = 2u + v\cos\theta \qquad v_{f_y} = v\sin\theta$$
$$v_f^2 = v_{f_x}^2 + v_{f_y}^2 = (2u + v\cos\theta)^2 + (v\sin\theta)^2$$
$$= 4u^2 + 4uv\cos\theta + v^2\cos^2\theta + v^2\sin^2\theta$$
$$= 4u^2 + 4uv\cos\theta + v^2$$
$$\therefore \boxed{v_f = \sqrt{4u^2 + v^2 + 4uv\cos\theta}}$$

(G1.4) Table on the last page gives data of Voyager-2 spacecraft for a few months in the year 1979 as it passed close to Jupiter. Assume that the observer is located at the centre of the Sun. The distance from the observer is given in AU and λ is heliocentric ecliptic longitude in degrees. Assume all objects to be in the ecliptic plane. Assume the orbit of the Earth to be circular. Plot appropriate column against the date of observation to find the date at which the spacecraft was closest to the Jupiter, and label the graph as G1.4. **8**

Solution:



From the graph, it can be inferred that the encounter with Jupiter occured on $\boxed{10^{\text{th}}\text{ July}}$ (day 191) and its distance from the Sun on that day is 5.33121 AU

(G1.5) Find the Earth-Jupiter distance, (d_{E-J}) on the day of the encounter. **4**

Solution:
The day number of Vernal Equinox is 80. Thus, ecliptic longitude of the Sun as seen from the Earth on the day of encounter will be,

$$\lambda_{\odot} = (191 - 80) * 360^\circ/365.25 = 109.4045^\circ$$

Thus, the ecliptic longitude of the Earth as seen from the Sun on the day of encounter will be,

$$\lambda_{\oplus} = 180^\circ + 109.4045^\circ = 289.4045^\circ$$

Applying cosine rule,

$$\begin{aligned} d_{\oplus - J} &= \sqrt{d_{\oplus}^2 + d_J^2 - 2 d_{\oplus} d_J \cos\Delta\lambda} \\ &= \sqrt{1^2 + 5.3312^2 - 2 \times 1 \times 5.3312 \times \cos(289.4045^\circ - 137.5628^\circ)} \\ &= 6.2308\,\text{AU} \end{aligned}$$

i.e. the Earth is $\boxed{6.2308\,\text{AU}}$ from Jupiter on that day.

(G1.6) On the day of the encounter, around what standard time (t_{std}) had the Jupiter transited the meridian in the sky of Bhubaneswar (20.27° N; 85.84° E; UT + 05:30)? **6**

Solution:
Thus, the angle of eastern elongation for Jupiter ($\measuredangle SEJ$) on that day would be,

$$\begin{aligned} \xi &= \sin^{-1}\left(\frac{5.3312 \times \sin(289.4045^\circ - 137.5628^\circ)}{6.2308}\right) \\ &= 23.8146^\circ \end{aligned}$$

It would rise 95 minutes after the Sun rise, i.e. around 7:35am. It would transit the meridian after around 6 hours i.e. around 13:35 local time or $\boxed{\text{13:22 IST}}$.
For more precise answer, R.A. of Jupiter on the day of encounter is approximately,

$$\begin{aligned} \lambda_{J_{\text{geocentric}}} &= 109.4045^\circ + 23.8146^\circ = 133.2191^\circ \\ \tan\alpha_J &= \tan\lambda_{J_{\text{geocentric}}} \cos\epsilon \\ &= \tan 133.2191^\circ \cos 23^\circ 26' \\ \therefore \alpha_J &= 135.68^\circ = 9^h\ 3^m \end{aligned}$$

Thus, it will culminate at that sidereal time. On that day, sidereal time at noon is 07:24 (111 days from V.E. times 4 minutes). Thus, it will culminate 1 hour 39 minutes after the local noon i.e. at 13:39 local time or at about $\boxed{\text{13:26 IST}}$.

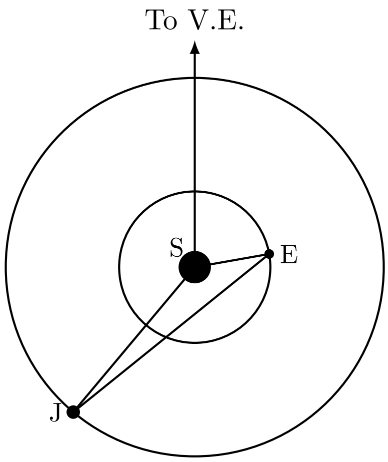


Figure 4

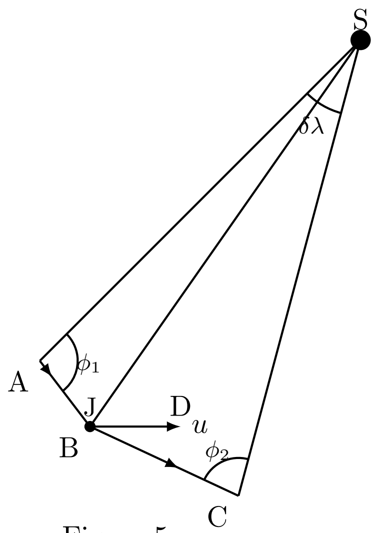


Figure 5

(G1.7) Speed of the spacecraft (in $\mathrm{km\,s^{-1}}$) as measured by the same observer on some dates before the encounter and some dates after the encounter are given below. Here day n is the date of encounter. Use these data to find the orbital speed of Jupiter (u) on the date of encounter and angle θ. **12**

date	$n-45$	$n-35$	$n-25$	$n-15$	$n-5$	n
$\mathbf{v}_{\text{tot}}$	10.1408	10.0187	9.9078	9.8389	10.2516	25.5150
date	$n+5$	$n+15$	$n+25$	$n+35$	$n+45$	
$\mathbf{v}_{\text{tot}}$	21.8636	21.7022	21.5580	21.3812	21.2365	

Solution:

In Figure 5, path of Voyager-2 is shown as A-B-C. The Sun is shown as S and the Jupiter is shown as J. From the data we note that r is increasing continuously. The same should be reflected in the diagram. For practical purpose, J and B are the same points. The direction of velocity vector of Jupiter is given by JD. In the figure,

$$\measuredangle ASB = \delta\lambda_1 \qquad \measuredangle ASB = \delta\lambda_2$$
$$\measuredangle ASC = \delta\lambda \qquad \measuredangle ABD = \theta$$
$$\measuredangle ABC = \theta_1 \qquad \measuredangle DBC = \measuredangle ABC - \measuredangle ABD = \theta_1 - \theta$$
$$\measuredangle SAB = \phi_1 \qquad \measuredangle SCB = \phi_2$$

Now the lines originating from the Sun indicate radial direction on the respective dates. Let us take speed of the spacecraft sufficiently far from the day 190, to avoid any influence of Jupiter in initial and final velocity estimation. We can choose dates 35 days on either side of July 10 i.e. June 5 and August 14.

$$\delta\lambda_1 = 137.5628^\circ - 136.0736^\circ = 1.4892^\circ$$
$$l(AB) = \sqrt{l(SA)^2 + l(SB)^2 - 2\times l(SA)\times l(SB)\times\cos\delta\lambda_1}$$
$$= \sqrt{5.17487^2 + 5.33121^2 - 2\times 5.17487\times 5.33121\times\cos 1.4892^\circ}$$
$$= 0.207\,55\,\mathrm{au}$$
$$\phi_1 = \sin^{-1}\left(\frac{l(SB)\sin\delta\lambda_1}{l(AB)}\right) = \sin^{-1}\left(\frac{5.33121\times\sin 1.4892^\circ}{0.20755}\right)$$

$$
\begin{aligned}
&= \sin^{-1}(0.66755)\\
\phi_1 &= 41.8783^\circ \text{ or } \ 138.1217^\circ\\
\delta\lambda_2 &= 141.2007^\circ - 137.5628^\circ = 3.6379^\circ\\
l(BC) &= \sqrt{l(SC)^2 + l(SB)^2 - 2\times l(SC)\times l(SB)\times\cos\delta\lambda_2}\\
&= \sqrt{5.45085^2 + 5.33121^2 - 2\times 5.45085\times 5.33121\times\cos 3.6379^\circ}\\
&= 0.362\,53\,\text{au}\\
\phi_2 &= \sin^{-1}\left(\frac{l(SB)\sin\delta\lambda_2}{l(BC)}\right) = \sin^{-1}\left(\frac{5.33121\times\sin 3.6379^\circ}{0.36253}\right)\\
&= \sin^{-1}(0.66755)\\
\phi_2 &= 68.9199^\circ \text{ or } \ 111.0801^\circ
\end{aligned}
$$

from the figure, ϕ_1 should be obtuse and ϕ_2 may be acute. In $\square SABC$

$$
\begin{aligned}
\delta\lambda &= \lambda_2 - \lambda_1 = 141.2007^\circ - 136.0736^\circ\\
&= 5.1271^\circ\\
\therefore \theta_1 &= 360^\circ - \delta\lambda - \phi_1 - \phi_2\\
&= 360^\circ - 5.1271^\circ - 138.1217^\circ - 68.9199^\circ\\
\theta_1 &= 147.8313^\circ
\end{aligned}
$$

In $\triangle SBC$, we notice

$$
\begin{aligned}
\measuredangle SBC &= 180^\circ - \phi_2 - \delta\lambda_2\\
&= 180^\circ - 68.9199^\circ - 3.6379^\circ\\
&= 107.4422^\circ\\
\tan\measuredangle DBC &= \frac{v_{y_f}}{v_{x_f}} = \frac{v\sin\theta}{v\cos\theta + 2u}\\
\tan(\theta_1 - \theta) &= \frac{\sin\theta}{\cos\theta + 2\frac{u}{v}}\\
\therefore \frac{2u}{v} &= \frac{\sin\theta}{\tan(\theta_1-\theta)} - \cos\theta
\end{aligned}
$$

We use this expression to find $|\vec{u}|$.

$$
\begin{aligned}
v_f^2 &= 4u^2 + v^2 + 4uv\cos\theta\\
\frac{v_f^2}{v^2} &= \frac{4u^2}{v^2} + 1 + \frac{4u}{v}\cos\theta\\
\left(\frac{v_f}{v}\right)^2 &= \left(\frac{2u}{v}\right)^2 + 1 + 2\left(\frac{2u}{v}\right)\cos\theta\\
&= \left(\frac{\sin\theta}{\tan(\theta_1-\theta)} - \cos\theta\right)^2 + 1 + 2\left(\frac{\sin\theta}{\tan(\theta_1-\theta)} - \cos\theta\right)\cos\theta\\
&= \left(\frac{\sin\theta}{\tan(\theta_1-\theta)}\right)^2 - \cancel{\frac{2\sin\theta\cos\theta}{\tan(\theta_1-\theta)}} + \cos^2\theta + 1 + \cancel{\frac{2\sin\theta\cos\theta}{\tan(\theta_1-\theta)}} - 2\cos^2\theta\\
&= \frac{\sin^2\theta}{\tan^2(\theta_1-\theta)} + 1 - \cos^2\theta\\
&= \frac{\sin^2\theta}{\tan^2(\theta_1-\theta)} + \sin^2\theta = \sin^2\theta\left(\cot^2(\theta_1-\theta)+1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sin^2\theta}{\sin^2(\theta_1-\theta)} \\
\therefore \frac{v_f}{v} &= \frac{\sin\theta}{\sin(\theta_1-\theta)} = \frac{\sin\theta}{\sin\theta_1\cos\theta - \cos\theta_1\sin\theta} \\
\therefore \frac{v}{v_f} &= \sin\theta_1\cot\theta - \cos\theta_1 \\
\tan\theta &= \frac{\sin\theta_1}{\frac{v}{v_f}+\cos\theta_1} \\
&= \frac{\sin 147.8313^\circ}{\frac{10.0187}{21.3812}+\cos 147.8313^\circ} = -1.4088 \\
\therefore \theta &= 180^\circ - 54.6328^\circ = 125.3672^\circ \\
v_f^2 &= 4u^2 + v^2 + 4uv\cos\theta \\
21.3812^2 &= 4u^2 + 10.0187^2 + 4u \times 10.0187\cos 125.3672^\circ \\
0 &= u^2 - 5.7990u - \frac{(457.1557-100.3743)}{4} \\
0 &= u^2 - 5.7990u - 89.1953 \\
\therefore u &= \frac{5.7990+\sqrt{5.7990^2+4\times 89.1953}}{2} \\
&= 12.7789\,\mathrm{km\,s^{-1}}
\end{aligned}
$$

Jupiter's orbital velocity on the day of encounter is $\boxed{12.779\,\mathrm{km\,s^{-1}}}$ and the angle between the initial velocity of the spacecraft and Jupiter's velocity vectors is $\boxed{125^\circ 22'}$.

(G1.8) Find eccentricity, e_J, of Jupiter's orbit. **8**

Solution:

The angle between $\vec{r}$ and $\vec{u}$ on the day of encounter will be,

$$
\begin{aligned}
\psi &= \measuredangle SBC - (\theta_1 - \theta) \\
&= 107.4422^\circ - 147.8313^\circ + 125.3672^\circ \\
&= 84.9781^\circ
\end{aligned}
$$

Now we use angular momentum conservation to estimate eccentricity. If u_p and r_p represent perihelion velocity and perihelion distance of Jupiter,

$$
\begin{aligned}
r_p u_p &= a_J(1-e)\sqrt{\frac{GM_\odot}{a_J}\left(\frac{1+e}{1-e}\right)} \\
&= \sqrt{GM_\odot a_J(1-e^2)} \\
r_p u_p &= ru\sin\psi \\
\therefore 1-e^2 &= \frac{r^2u^2\sin^2\psi}{GM_\odot a_J} \\
&= \frac{5.33121^2 \times 1.496\times 10^{11} \times \left(12.7789\times 10^3\right)^2 \sin^2 84.9781^\circ}{6.6741\times 10^{-11} \times 1.9891\times 10^{30} \times 5.202\,60} \\
&= 0.99761 \\
\therefore e &= \sqrt{1-0.99761} = 0.0489
\end{aligned}
$$

The eccentricity of Jupiter's orbit is $\boxed{0.0489}$.

(G1.9) Find heliocentric ecliptic longitude, λ_p, of Jupiter's perihelion point. **5**

Solution:

To estimate longitude of perihelion, one should estimate true anomaly of Jupiter on that day.

$$
\begin{aligned}
r &= \frac{a(1-e^2)}{1+e\cos\Theta} \\
\therefore 0.0489\cos\Theta &= \frac{a(1-e^2)}{r} - 1 = \frac{5.20260 \times 0.99761}{5.33121} - 1 \\
&= -0.02646 \\
\Theta &= 122.754^\circ
\end{aligned}
$$

Thus, the longitude of perihelion of Jupiter is,

$$
\begin{aligned}
\lambda_p &= \lambda_J - \Theta \\
&= 137.5628^\circ - 122.754^\circ \\
\lambda_p &= 14.809^\circ
\end{aligned}
$$

Month	Date	λ (°)	Distance (AU)
June	1	135.8870	5.1589731906
June	2	135.9339	5.1629499712
June	3	135.9806	5.1669246607
June	4	136.0272	5.1708975373
June	5	136.0736	5.1748689006
June	6	136.1200	5.1788390741
June	7	136.1662	5.1828084082
June	8	136.2122	5.1867772826
June	9	136.2582	5.1907461105
June	10	136.3040	5.1947153428
June	11	136.3496	5.1986854723
June	12	136.3951	5.2026570402
June	13	136.4405	5.2066306418
June	14	136.4857	5.2106069354
June	15	136.5307	5.2145866506
June	16	136.5756	5.2185705999
June	17	136.6202	5.2225596924
June	18	136.6647	5.2265549493
June	19	136.7090	5.2305575243
June	20	136.7532	5.2345687280
June	21	136.7970	5.2385900582
June	22	136.8407	5.2426232385
June	23	136.8841	5.2466702671
June	24	136.9273	5.2507334797
June	25	136.9702	5.2548156324
June	26	137.0127	5.2589200110
June	27	137.0550	5.2630505798
June	28	137.0969	5.2672121872
June	29	137.1384	5.2714108557
June	30	137.1795	5.2756542053
July	1	137.2200	5.2799520895
July	2	137.2600	5.2843175880
July	3	137.2993	5.2887686308
July	4	137.3378	5.2933308160
July	5	137.3754	5.2980426654
July	6	137.4118	5.3029664212
July	7	137.4467	5.3082133835
July	8	137.4798	5.3140161793
July	9	137.5116	5.3210070441
July	10	137.5628	5.3312091210
July	11	137.6898	5.3405592121
July	12	137.8266	5.3466522674
July	13	137.9599	5.3516661563
July	14	138.0903	5.3561848203
July	15	138.2186	5.3604205657
July	16	138.3453	5.3644742164

Month	Date	λ (°)	Distance (AU)
July	17	138.4707	5.3684017790
July	18	138.5949	5.3722377051
July	19	138.7183	5.3760047603
July	20	138.8409	5.3797188059
July	21	138.9628	5.3833913528
July	22	139.0841	5.3870310297
July	23	139.2048	5.390644477
July	24	139.3250	5.3942369174
July	25	139.4448	5.3978125344
July	26	139.5641	5.4013747321
July	27	139.6831	5.4049263181
July	28	139.8016	5.4084696349
July	29	139.9198	5.4120066575
July	30	140.0377	5.4155390662
July	31	140.1553	5.4190683021
August	1	140.2725	5.4225956100
August	2	140.3895	5.4261220723
August	3	140.5062	5.4296486357
August	4	140.6225	5.4331761326
August	5	140.7387	5.4367052982
August	6	140.8546	5.4402367851
August	7	140.9702	5.4437711745
August	8	141.0856	5.4473089863
August	9	141.2007	5.4508506867
August	10	141.3157	5.4543966955
August	11	141.4303	5.4579473912
August	12	141.5448	5.4615031166
August	13	141.6591	5.4650641822
August	14	141.7731	5.4686308707
August	15	141.8869	5.4722034391
August	16	142.0006	5.4757821220
August	17	142.1140	5.4793671340
August	18	142.2272	5.4829586711
August	19	142.3402	5.4865569133
August	20	142.4530	5.4901620256
August	21	142.5657	5.4937741595
August	22	142.6781	5.4973934544
August	23	142.7904	5.5010200385
August	24	142.9024	5.5046540300
August	25	143.0143	5.5082955377
August	26	143.1260	5.5119446617
August	27	143.2375	5.5156014948
August	28	143.3488	5.5192661222
August	29	143.4599	5.5229386226
August	30	143.5709	5.5266190687
August	31	143.6817	5.5303075275